Matchings

A matching is a set of (non-loop) edges with no shared endpoints. The vertices incident to an edge of a matching M are saturated by M, the others are unsaturated. A perfect matching of G is matching which saturates all the vertices.

Examples. $K_{n,m}, K_n$, Petersen graph, Q_k; graphs without perfect matching

A maximal matching cannot be enlarged by adding another edge.

A maximum matching of G is one of maximum size.

Example. Maximum $\neq$ Maximal

Characterization of maximum matchings

Let M be a matching. A path that alternates between edges in M and edges not in M is called an M-alternating path. An M-alternating path whose endpoints are unsaturated by M is called an M-augmenting path.

Theorem(Berge, 1957) A matching M is a maximum matching of graph G iff G has no M-augmenting path.

Lemma Let M_1 and M_2 be matchings of G. Then each connected component of $M_1 \triangle M_2$ is a path or an even cycle.

For two sets A and B, the symmetric difference is $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Hall's Condition and consequences

Theorem (Hall, 1935) Let G be a bipartite graph with partite sets X and Y. Then there is a matching in G saturating X iff $|N(S)| \geq |S|$ for every $S \subseteq X$.

Proof. ($\Longrightarrow$) Easy.

($\Longleftarrow$) Not *so* easy. (Find an M-augmenting path for every matching M which does not saturate X.)

Corollary. (Marriage Theorem, Frobenius (1917)) For $k > 0$, every k-regular bipartite graph has a perfect matching.

Graph parameters — Definitions and simple properties

The size of the largest independent set in G is denoted by $\alpha(G)$.

The size of the largest matching (*independent set of edges*) in G is denoted by $\alpha'(G)$.

A vertex cover of G is a set $Q \subseteq V(G)$ that contains at least one endpoint of every edge. (The vertices in Q *cover* $E(G)$).
The size of the smallest vertex cover in G is denoted by $\beta(G)$.

Claim. $\beta(G) \geq \alpha'(G)$.

An edge cover of G is a set L of edges such that every vertex of G is incident to some edge in L.
The size of the smallest edge cover in G is denoted by $\beta'(G)$.

Claim. $\beta'(G) \geq \alpha(G)$.

Min-max theorems for bipartite graphs

Theorem. (König (1931), Egerváry (1931)) If G is bipartite then $\beta(G) = \alpha'(G)$.

Proof. For any minimum vertex cover Q, apply Hall's Condition to match $Q \cap X$ into $Y \setminus Q$ and $Q \cap Y$ into $X \setminus Q$.

Lemma. Let G be any graph. $S \subseteq V(G)$ is an independent set iff $\overline{S}$ is a vertex cover. Hence $\alpha(G) + \beta(G) = n(G)$.

Proof. Easy.

Theorem. (Gallai, 1959) Let G be any graph without isolated vertices. Then $\alpha'(G) + \beta'(G) = n(G)$.

Corollary. (König, 1916) Let G be a bipartite graph with no isolated vertices. Then $\alpha(G) = \beta'(G)$.

Proof. Put together the previous three statements.