Connectivity

A separating set (or vertex cut) of a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more than one component. For $G \neq K_n$, the connectivity of G is $\kappa(G) := min\{|S| : S \text{ is a vertex cut}\}$. By definition, $\kappa(K_n) := n - 1$. A graph G is k-connected if there is no vertex cut of size $k - 1$. (i.e. $\kappa(G) \geq k$)

Examples. $\kappa(K_{n,m}) = \min\{n, m\}$
$\kappa(Q_n) = n$

Extremal problem: What is the minimum number of edges in a k-connected graph?

Theorem. For every n, the minimum number of edges in a k-connected graph is $\lceil kn/2 \rceil$.

Proof:

min $\geq \lceil kn/2 \rceil$, since $k \leq \kappa(G) \leq \delta(G)$
min $\leq \lceil kn/2 \rceil$; Example: Harary graphs $H_{k,n}$.

Edge-connectivity

A disconnecting set of a graph G is a set $F \subseteq E(G)$ of edges such that $G - F$ has more than one component. The edge-connectivity of G is

$$\kappa'(G) := \min\{|F| : F \text{ is a disconncting set}\}.$$

A graph G is k-edge-connected if there is no disconnecting set of size $k - 1$ (i.e. $\kappa'(G) \geq k$).
An edge cut is an edge-set of the form $[S, \bar{S}]$, where $\emptyset \neq S \neq V(G)$ and $\bar{S} = V(G) \setminus S$.
For $S, T \subseteq V(G)$, $[S, T] := \{xy \in E(G) : x \in S, y \in T\}$.

Implications. edge cut $\Rightarrow$ disconnecting set
edge cut $\not\Leftarrow$ disconnecting set
edge cut $\Leftarrow$ *minimal* disconnecting set

Theorem. (Whitney, 1932) If G is a simple graph, then $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

Homework. Example of a graph G with $\kappa(G) = k$, $\kappa'(G) = l$, $\delta(G) = m$, for any $0 < k \leq l \leq m$.

Theorem. G is 3-regular $\Rightarrow$ $\kappa(G) = \kappa'(G)$.

Characterization of 2-connected graphs

Theorem. (Whitney,1932) Let G be a graph, $n(G) \geq 3$. Then G is 2-connected iff for every $u, v \in V(G)$ there exist two internally disjoint u, v-paths in G.

Theorem. Let G be a graph with $n(G) \geq 3$. The the following four statements are equivalent.

(i) G is connected and has no cut-vertex
(ii) For all $x, y \in V(G)$, there are two internally disjoint x, y-path.
(iii) For all $x, y \in V(G)$, there is a cycle through x and y.
(iv) $\delta(G) \geq 1$, and every pair of edges of G lies on a common cycle.

Expansion Lemma. Let G' be a supergraph of a k-connected graph G obtained by adding one vertex to $V(G)$ with at least k neighbors.
Then G' is k-connected as well.

(Closed-)Ear Decompositions

An ear of a graph G is a maximal path whose internal vertices have degree 2 in G. An ear decomposition of G is a decomposition $P_0, \ldots, P_k$ such that P_0 is a cycle and P_i, for $i \geq 1$ is an ear of $P_0 \cup \ldots \cup P_i$.

Theorem. (Whitney, 1932) A graph G is 2-connected iff it has an ear-decomposition.

In a graph G, subdivision of an edge uv is the operation of replacing uv with a path u, w, v through a new vertex w.
Claim. If G is 2-connected, then the graph G' obtained by subdividing an edge of G is 2-connected.

A closed ear of a graph G is a cycle C such that all vertices of C except one have degree 2 in G. A closed-ear decomposition of G is a decomposition P_0, $P_1, \ldots, P_k$ such that P_0 is a cycle and P_i, for $i \geq 1$ is either an (open) ear or closed ear in $P_0 \cup \ldots \cup P_i$.

Theorem. A graph G is 2-edge-connected iff it has a closed-ear decomposition.

Remark. In the previous two theorems one can choose *any cycle* as a starting cycle of a (closed-)ear decomposition.

Menger's Theorem

Given $x, y \in V(G)$, a set $S \subseteq V(G) \setminus \{x, y\}$ is an x, y-separator (or an x, y-cut) if $G - S$ has no x, y-path.
A set $\mathcal{P}$ of paths is called pairwise internally disjoint (p.i.d.) if for any two path $P_1, P_2 \in \mathcal{P}$, P_1 and P_2 have no common internal vertices.
Define

$\kappa(x, y)$:= min$\{|S| : S$ is an x, y-cut,$\}$ and
$\lambda(x, y)$:= max$\{|\mathcal{P}| : \mathcal{P}$ is a set of p.i.d. x, y-paths$\}$

Theorem (Menger, 1927) Let $x, y \in V(G)$, such that $xy \notin E(G)$. Then $\kappa(x, y) = \lambda(x, y)$.

Corollary
$\kappa(G) = \max\{k : \lambda(x, y) \geq k$ for all $x, y \in V(G)\}$.

Proof: **Lemma.** For every $e \in E(G)$, $\kappa(G - e) \geq \kappa(G) - 1$.

Proof of $\lambda(x, y) \geq \kappa(x, y)$

Induction on $n(G)$.
Case 1. G has a minimum x, y-cut S, $S \neq N(x)$ and $S \neq N(y)$; $|S| = \kappa(x, y) = k$.

Proof. Combine k x, S-paths with k S, y-paths.
They exist, because of induction.

$V_1 := \{w \in V(G) : w$ is on a x, S-path$\}$
$V_2 := \{w \in V(G) : w$ is on an S, y-path$\}$
Claim. $S = V_1 \cap V_2$
Define graph H_1 on $V_1 \cup \{y'\}$ and use induction.
Similarly for graph H_2 on $V_2 \cup \{x'\}$.

Case 2. Every minimum x, y-cut is either $N(x)$ or $N(y)$.

Case 2.a. There is a $v \notin N(x) \cup N(y) \cup \{x, y\}$.
Proof: Then $\kappa_{G-v}(x, y) = \kappa_G(x, y)$.
By induction $\kappa_{G-v}(x, y) = \lambda_{G-v}(x, y)$.

Case 2.b. There is a $u \in N(x) \cap N(y)$.
Proof: Then $\kappa_{G-u}(x, y) = \kappa_G(x, y) - 1$.
By induction $\kappa_{G-u}(x, y) = \lambda_{G-u}(x, y)$.

Case 2.c. $V(G) = N(x) \cup N(y) \cup \{x, y\}$.
Proof: Apply the König-Egervary Theorem.

Edge-Menger

Given $x, y \in V(G)$, a set $F \subseteq E(G)$ is an x, y-disconnecting set if $G - F$ has no x, y-path. Define

$\kappa'(x, y)$:= min$\{|F| : F$ is an x, y-disconnecting set,$\}$
$\lambda'(x, y)$:= max$\{|\mathcal{P}| : \mathcal{P}$ is a set of p.e.d.* x, y-paths$\}$

* p.e.d. means pairwise edge-disjoint

Theorem For all distinct $x, y \in V(G)$,

$$\kappa'(x, y) = \lambda'(x, y).$$

Proof. Apply Menger's Theorem for the line graph of G', where $V(G') = V(G) \cup \{s, t\}$ and $E(G') = E(G) \cup \{sx, yt\}$.

The line graph $L(G)$ of a graph G is defined by
$V(L(G)) := E(G)$,
$E(L(G)) := \{ef : e$ and f share an endpoint$\}$.

Corollary
$\kappa'(G) = \max\{k : \lambda'(x, y) \geq k$ for all $x, y \in V(G)\}$.