Vertex coloring, chromatic number*

A k-coloring of a graph G is a labeling $f : V(G) \to S$, where $|S| = k$. The labels are called colors; the vertices of one color form a color class. A k-coloring is proper if adjacent vertices have different labels. A graph is k-colorable if it has a proper k-coloring. The chromatic number $\chi(G) := \min\{k : G \text{ is } k\text{-colorable}\}$. A graph G is k-chromatic if $\chi(G) = k$. A proper k-coloring of a k-chromatic graph is an optimal coloring.

Examples. K_n, $K_{n,m}$, C_5, Petersen

A graph is G is k-color-critical (or k-critical) if $\chi(H) < \chi(G) = k$ for every *proper* subgraph H of G.

Characterization of 1-, 2-, 3-critical graphs.

*All graphs are loopless

Lower bounds

Simple lower bounds

$$\chi(G) \geq \omega(G)$$
$$\chi(G) \geq \frac{n(G)}{\alpha(G)}$$

Examples for $\chi(G) \neq \omega(G)$: $C_5 \vee K_s$.

Examples for $\chi(G) = \omega(G)$: cliques, *interval graphs*, *perfect graphs*.

An interval representation of a graph is an assignment of an interval to the vertices of the graph, such that two vertices are adjacent iff the corresponding intervals intersect. A graph having such a representation is called an interval graph.

Proposition. If G is an interval graph, then $\chi(G) = \omega(G)$.

Proof. Order vertices according to left endpoints of corresponding intervals and color *greedily*.

Upper bounds

Proposition $\chi(G) \leq \Delta(G) + 1$.

Proof. Algorithmic; Greedy coloring.

A graph G is d-degenerate if every subgraph of G has minimum degree at most d.

Claim. G is d-degenerate iff there is an ordering of the vertices $v_1, \ldots, v_n$, such that $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}|$

Proposition. For a d-degenerate G, $\chi(G) \leq d + 1$.
In particular, for every G, $\chi(G) \leq \max_{H \subseteq G} \delta(H) + 1$.

Proof. Greedy coloring.

Brooks' Theorem. (1941) Let G be a connected graph. Then $\chi(G) = \Delta(G) + 1$ iff G is a complete graph or an odd cycle.

Mycielski's Construction

The bound $\chi(G) \geq \omega(G)$ could be arbitrarily bad.

Construction. Given graph G with vertices $v_1, \ldots, v_n$, we define supergraph $M(G)$.

$V(M(G)) = V(G) \cup \{u_1, \ldots u_n, w\}$.

$E(M(G)) = E(G) \cup \{u_i v : v \in N_G(v_i) \cup \{w\}\}$.

Theorem. If G is triangle-free, then so is $M(G)$.
If $\chi(G) = k$, then $\chi(M(G)) = k + 1$.

Forced subdivision

G contains a $K_k \Rightarrow \chi(G) \geq k$

G contains a $K_k \not\Leftarrow \chi(G) \geq k$ (already for $k \geq 3$)

Hajós' Conjecture
G contains a K_k-subdivision $\overset{?}{\Leftarrow} \chi(G) \geq k$

An H-subdivision is a graph obtained from H by successive edge-subdivisions.

Remark. The conjecture is true for $k = 2$ and $k = 3$.

Theorem (Dirac, 1952) Hajós' Conjecture is true for $k = 4$.

Homework. Hajós' Conjecture is false for $k \geq 7$.

Hadwiger's Conjecture
G contains a K_k*-minor* $\overset{?}{\Leftarrow} \chi(G) \geq k$

Proved for $k \leq 6$. Open for $k \geq 7$.

Proof of Dirac's Theorem

Theorem (Dirac, 1952) If $\chi(G) \geq 4$ then G contains a K_4-subdivision.

Proof. Induction on $n(G)$. $n(G) = 4 \Rightarrow G = K_4$.

W.l.o.g. G is 4-critical.

Case 0. $\kappa(G) = 0$ would contradict 4-criticality.

Case 1. $\kappa(G) = 1$ would contradict 4-criticality.

Case 2. $\kappa(G) = 2$. Let $S = \{x, y\}$ be a cut-set.

$xy \in E(G)$ would contradict 4-criticality.
Hence $xy \notin E(G)$.

By 4-criticality, G must have an S-lobe H, such that $\chi(H + xy) \geq 4$. Apply induction hypothesis to $H + xy$ and find a K_4-subdivision in $H + xy$. Then modify F to obtain a K_4-subdivision in G.

Let $S \subseteq V(G)$. An S-lobe of G is an induced subgraph of G whose vertex set consists of S and the vertices of a component of $G - S$.

Proof of Dirac's Theorem— Continued

Case 3. $\kappa(G) \geq 3$. Let $x \in V(G)$. $G - x$ is 2-connected, so contains a cycle C of length at least 3.

Claim. There is an x, C-fan of size 3.

Proof. Add a new vertex u to G connecting it to the vertices of C. By the Expansion Lemma the new graph G' is 3-connected. By Menger's Theorem there exist three p.i.d x, u-paths P_1, P_2, P_3 in G'. $\square$

Given a vertex x and a set U of vertices, and x, U-fan is a set of paths from x to U such that any two of them share only the vertex x.
Fan Lemma. G is k-connected iff $|V(G)| \geq k + 1$ and for every choice of $x \in V(G)$ and $U \subseteq V(G)$, G has an x, U-fan.

Then $C \cup P_1 \cup P_2 \cup P_3 - u$ is K_4-subdivision in G.

Complete k-partite graphs

G is a complete k-partite graph if there is a partition $V_1 \cup \ldots V_k = V(G)$ of the vertex set, such that $uv \in E(G)$ iff u and v are in *different* parts of the partition. If $|V_i| = n_i$, then G is denoted by $K_{n_1,\ldots,n_k}$.

The Turán graph $T_{n,r}$ is the complete r-partite graph on n vertices whose partite sets differ in size by at most 1. (All partite sets have size $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$.)

Lemma Among r-colorable graphs the Turán graph is the *unique* graph, which has the most number of edges.

Turán's Theorem

The Turán number $ex(n, H)$ of a graph H is the largest integer m such that there exists an H-free graph on n vertices with m edges.

Example: Mantel's Theorem states $ex(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$.

Theorem. (Turán, 1941)

$$ex(n, K_r) = e(T_{n,r-1}) = \left(1 - \frac{1}{r-1} + O\left(\frac{1}{n}\right)\right)\binom{n}{2}.$$

Proof. Prove by induction on r that if $G \not\supseteq K_r$ then there is an $(r-1)$-partite graph H with $V(H) = V(G)$ and $e(H) \geq e(G)$. Then use the Lemma to finish the proof.

Turán-type problems

Question. (Turán, 1941) What happens if instead of K_4, which is the graph of the tetrahedron, we forbid the graph of some other platonic polyhedra? How many edges can a graph without an octahedron (or cube, or dodecahedron or icosahedron) have?

Theorem. (Erdős, Simonovits, Stone) For any graph H,

$$ex(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right)\binom{n}{2} + o(n^2).$$

Corollaries.

$$\begin{aligned} ex(n, \texttt{octahedron}) &= \frac{n^2}{4} + o(n^2) \\ ex(n, \texttt{dodechedron}) &= \frac{n^2}{4} + o(n^2) \\ ex(n, \texttt{icosahedron}) &= \frac{n^2}{3} + o(n^2) \end{aligned}$$

Open problems and Conjectures

Known results.

$$\begin{aligned} \Omega(n^{3/2}) &\leq ex(n, Q_3) &\leq O(n^{8/5}) \\ \Omega(n^{1/8}) &\leq ex(n, C_8) &\leq O(n^{1/4}) \\ \Omega(n^{2/3}) &\leq ex(n, K_{4,4}) &\leq O(n^{7/4}) \end{aligned}$$

Conjectures.

$$\begin{aligned} ex(n, K_{t,s}) &= \Theta\left(n^{2 - \frac{1}{\min\{t,s\}}}\right) \\ ex(n, C_{2k}) &= \Theta\left(n^{\frac{1}{k}}\right) \\ ex(n, Q_3) &= \Theta\left(n^{\frac{8}{5}}\right) \end{aligned}$$

If H is a d-degenerate bipartite graph, then

$$ex(n, H) = O\left(n^{2 - \frac{1}{t}}\right).$$