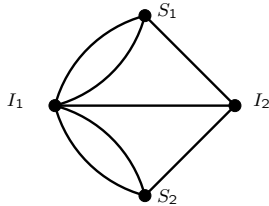
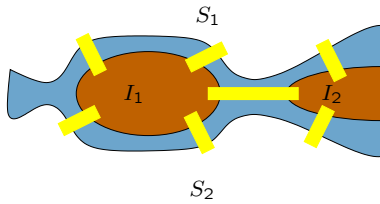


Graph Theory



1

Graphs – Definition_____

A **graph** G is a pair consisting of

- a vertex set $V(G)$, and
- an edge set $E(G) \subseteq \binom{V(G)}{2}$.

x and y are the **endpoints** of edge $e = \{x, y\}$.

They are called **adjacent** or **neighbors**.

e is called **incident** with x and y .

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Multigraphs: Extension & Confusion_____

A **loop** is an edge whose endpoints are equal.

Multiple edges are edges having the same set of endpoints.

Our book allows both loops and multiple edges in “graphs”. We don’t – at least when we say “graph”. When we do want to allow multiple edges or loops we say **multigraph**. When the book wants to talk about a graph without multiple edges and loops, it says **simple graph**.*

Remarks A multigraph might have no multiple edges or loops. Every (simple) graph is a multigraph, but not every multigraph is a (simple) graph.

Every graph is finite.†

*Sometimes even we say “simple graph”, when we would like to emphasize that there are no multiple edges and loops.

†in this course

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Special graphs_____

K_n is the complete graph on n vertices.

$K_{n,m}$ is the complete bipartite graph with partite sets of sizes n and m .

P_n is the path on n vertices

C_n is the cycle on n vertices

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Further definitions

The **degree** of vertex v is the number of edges incident with v . Loops are counted twice.

A set of pairwise adjacent vertices in a graph is called a **clique**. A set of pairwise non-adjacent vertices in a graph is called an **independent set**.

A graph G is **bipartite** if $V(G)$ is the union of two (possibly empty) independent sets of G . These two sets are called the **partite sets** of G .

The **complement** \overline{G} of a graph G is a graph with

- vertex set $V(\overline{G}) = V(G)$ and
- edge set $E(\overline{G}) = \binom{V}{2} \setminus E(G)$.

H is a **subgraph** of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$. We write $H \subseteq G$. We also say G **contains** H and write $G \supseteq H$.

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The Petersen graph

$$V(P) = \binom{[5]}{2}$$

$$E(P) = \{\{A, B\} : A \cap B = \emptyset\}$$

Properties.

- each vertex has degree 3 (i.e. P is 3-regular)
- adjacent vertices have no common neighbor
- non-adjacent vertices have exactly one common neighbor

Corollary. The girth of the Petersen graph is 5.

The **girth** of a graph is the length of its shortest cycle.

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Isomorphism of graphs

An **isomorphism** of G to H is a **bijection** $f : V(G) \rightarrow V(H)$ such that $uv \in E(G)$ iff* $f(u)f(v) \in E(H)$. If there is an isomorphism from G to H , then we say G is **isomorphic to** H , denoted by $G \cong H$.

Claim. The isomorphism relation is an equivalence relation on the set of all graphs.

An **isomorphism class** of graphs is an equivalence class of graphs under the isomorphism relation.

Example. What are those graphs for which the adjacency relation is an equivalence relation?

Remark. labeled vs. unlabeled

“unlabeled graph” \approx “isomorphism class”.

Example. What is the number of labeled and unlabeled graphs on n vertices?

*if and only if

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Equivalence relation

A **relation** on a set S is a subset of $S \times S$.

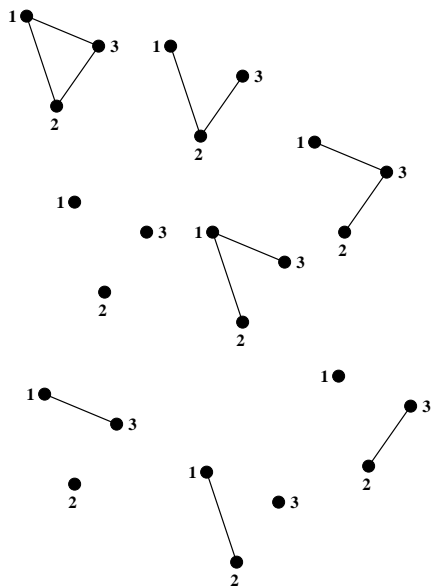
A relation R on a set S is an **equivalence relation** if

1. $(x, x) \in R$ (R is **reflexive**)
2. $(x, y) \in R$ implies $(y, x) \in R$ (R is **symmetric**)
3. $(x, y) \in R$ and $(y, z) \in R$ imply $(x, z) \in R$ (R is **transitive**)

An equivalence relation defines a **partition** of the base set S into **equivalence classes**. Elements are in relation iff they are within the same class.

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Isomorphism classes



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Automorphisms

An **automorphism** of G is an isomorphism of G to G . A graph G is **vertex transitive** if for every pair of vertices u, v there is an automorphism that maps u to v .

Examples.

- Automorphisms of P_4
- Automorphisms of $K_{r,s}$
- Automorphisms of Petersen graph.

A **decomposition** of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list.

A graph is **self-complementary** if it is isomorphic to its complement.

Example. P_4, C_5

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Walks, trails, paths, and cycles

A **walk** is an alternating list $v_0, e_1, v_1, e_2, \dots, e_k, v_k$ of vertices and edges such that for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i .

A **trail** is a walk with no repeated edge.

A **path** is a walk with no repeated vertex.

A u, v -walk, u, v -trail, u, v -path is a walk, trail, path, respectively, with first vertex u and last vertex v .

If $u = v$ then the u, v -walk and u, v -trail is **closed**. A closed trail (without specifying the first vertex) is a **circuit**. A circuit with no repeated vertex is called a **cycle**.

The **length** of a walk trail, path or cycle is its number of edges.

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Connectivity

G is **connected**, if there is a u, v -path for every pair $u, v \in V(G)$ of vertices. Otherwise G is **disconnected**.

Vertex u is **connected to** vertex v in G if there is a u, v -path. The **connection relation** on $V(G)$ consists of the ordered pairs (u, v) such that u is connected to v .

Claim. The connection relation is an equivalence relation.

Lemma. Every u, v -walk contains a u, v -path.

The **connected components** of G are its maximal connected subgraphs (i.e. the equivalence classes of the connection relation).

An **isolated vertex** is a vertex of degree 0. It is a connected component on its own, called **trivial** connected component.

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Theorem 1. (Principle of Induction) Let $P(n)$ be a statement with integer parameter n . If the following two conditions hold then $P(n)$ is true for each positive integer n .

1. $P(1)$ is true.
2. For all $n > 1$, " $P(n - 1)$ is true" implies " $P(n)$ is true".

Theorem 2. (Strong Principle of Induction) Let $P(n)$ be a statement with integer parameter n . If the following two conditions hold then $P(n)$ is true for each positive integer n .

1. $P(1)$ is true.
2. For all $n > 1$, " $P(k)$ is true for $1 \leq k < n$ " implies " $P(n)$ is true".

A **cut-edge** or **cut-vertex** of G is an edge or a vertex whose deletion increases the number of components.

If $M \subseteq E(G)$, then $G - M$ denotes the graph obtained from G by the deletion of the elements of M ; $V(G - M) = V(G)$ and $E(G - M) = E(G) \setminus M$. Similarly, for $S \subseteq V(G)$, $G - S$ obtained from G by the deletion of S and all edges incident with a vertex from S .

For $e \in E(G)$, $G - \{e\}$ is abbreviated by $G - e$.

For $v \in V(G)$, $G - \{v\}$ is abbreviated by $G - v$.

Proposition. An edge e is a cut-edge iff it does not belong to a cycle.