Extremal problems— Examples

Proposition 1. If G is an n-vertex graph with at most $n-2$ edges then G is disconnected.

Proof. By induction on $e(G)$ prove that every graph G has at least $n(G) - e(G)$ components.

A **Question** you always have to ask:
Can we improve on this proposition?

Answer. NO! The same statement is FALSE with $n-1$ in the place of $n-2$.
Proposition 1 is *best possible*, as shown by P_n.

Proposition 2. If G is an n-vertex graph with at least n edges then G contains a cycle.

Remark. Proposition 2 is also *best possible*, (e.g. P_n).

Proposition 1. + Remark: The minimum value of $e(G)$ over connected graphs is $n-1$.

Proposition 2. + Remark: The maximum value of $e(G)$ over acyclic (i.e. cycle-free) graphs is $n-1$.

Extremal problems — More example

Vague description: An extremal problem asks for the maximum or minimum value of a parameter over a class of objects (graphs, in most cases).

Proposition. G is an n-vertex graph with $\delta(G) \geq \lfloor n/2 \rfloor$, then G is connected.

Remark. The above proposition is *best possible*, as shown by $K_{\lfloor n/2 \rfloor} + K_{\lceil n/2 \rceil}$.

> Graph $G + H$ is the disjoint union (or sum) of graphs G and H. For an integer m, mG is the graph consisting of m disjoint copies of G.

Prop. + Remark: The maximum value of $\delta(G)$ over disconnected graphs is $\lfloor \frac{n}{2} \rfloor - 1$.

Extremal Problems

graph property	graph parameter	type of extremum	value of extremum
connected	$e(G)$	minimum	$n-1$
acyclic	$e(G)$	maximum	$n-1$
disconnected	$\delta(G)$	maximum	$\left\lfloor \frac{n}{2} \right\rfloor - 1$
K_3-free	$e(G)$	maximum	$\left\lfloor \frac{n^2}{4} \right\rfloor$

Triangle-free subgraphs

Theorem. (Mantel, 1907) The maximum number of edges in an n-vertex triangle-free graph is $\lfloor \frac{n^2}{4} \rfloor$.

Proof.

(i) *There is* a triangle-free graph with $\lfloor \frac{n^2}{4} \rfloor$ edges.

(ii) If G is a triangle-free graph, then $e(G) \leq \lfloor \frac{n^2}{4} \rfloor$.

Proof of (ii) is with extremality. (Look at the neighborhood of a vertex of maximum degree.)

Example of a wrong proof of (ii) by induction.

Bipartite subgraphs

Theorem. Every loopless multigraph G has a bipartite subgraph with at least $e(G)/2$ edges.

Proof # 1. Algorithmic. (Start from an arbitrary bipartition and move over a vertex whose degree in its own part is *more* than its degree in the other part. Iterate. Prove termination. Prove that at termination you have what you want.)

Proof # 2. Extremality. (Consider a bipartite subgraph H with the *maximum number of edges*, prove that $d_H(v) \geq d_G(v)/2$ for every vertex $v \in V(G)$ and use the Handshaking Lemma.)

Remark 1. *Maximum vs. maximal.* Algorithmic proof *not* necessarily ends up in bipartite subgraph with maximum number of edges.

Remark 2. The constant multiplier $\frac{1}{2}$ of $e(G)$ in the Theorem is best possible. *Example:* K_n.