

## Extremal problems— Examples

**Proposition 1.** If  $G$  is an  $n$ -vertex graph with **at most**  $n - 2$  edges then  $G$  is disconnected.

*Proof.* By induction on  $e(G)$  prove that every graph  $G$  has at least  $n(G) - e(G)$  components.

**A Question you always have to ask:**

Can we improve on this proposition?

**Answer.** NO! The same statement is **FALSE** with  $n - 1$  in the place of  $n - 2$ .

Proposition 1 is **best possible**, as shown by  $P_n$ .

**Proposition 2.** If  $G$  is an  $n$ -vertex graph with **at least**  $n$  edges then  $G$  contains a cycle.

**Remark.** Proposition 2 is also **best possible**, (e.g.  $P_n$ ).

**Proposition 1. + Remark:** The **minimum** value of  $e(G)$  over connected graphs is  $n - 1$ .

**Proposition 2. + Remark:** The **maximum** value of  $e(G)$  over acyclic (i.e. cycle-free) graphs is  $n - 1$ .

1

## Extremal problems — More example

Vague description: An **extremal problem** asks for the maximum or minimum value of a parameter over a class of objects (graphs, in most cases).

**Proposition.**  $G$  is an  $n$ -vertex graph with  $\delta(G) \geq \lfloor n/2 \rfloor$ , then  $G$  is connected.

**Remark.** The above proposition is **best possible**, as shown by  $K_{\lfloor n/2 \rfloor} + K_{\lceil n/2 \rceil}$ .

Graph  $G + H$  is the **disjoint union** (or **sum**) of graphs  $G$  and  $H$ . For an integer  $m$ ,  $mG$  is the graph consisting of  $m$  disjoint copies of  $G$ .

**Prop. + Remark:** The **maximum** value of  $\delta(G)$  over disconnected graphs is  $\lfloor \frac{n}{2} \rfloor - 1$ .

2

## Extremal Problems

graph property	graph parameter	type of extremum	value of extremum
connected	$e(G)$	minimum	$n - 1$
acyclic	$e(G)$	maximum	$n - 1$
disconnected	$\delta(G)$	maximum	$\lfloor \frac{n}{2} \rfloor - 1$
$K_3$ -free	$e(G)$	maximum	$\lfloor \frac{n^2}{4} \rfloor$

3

## Triangle-free subgraphs

**Theorem.** (Mantel, 1907) The maximum number of edges in an  $n$ -vertex **triangle-free** graph is  $\lfloor \frac{n^2}{4} \rfloor$ .

*Proof.*

(i) There is a triangle-free graph with  $\lfloor \frac{n^2}{4} \rfloor$  edges.

(ii) If  $G$  is a triangle-free graph, then  $e(G) \leq \lfloor \frac{n^2}{4} \rfloor$ .

Proof of (ii) is with extremality. (Look at the neighborhood of a vertex of maximum degree.)

*Example* of a wrong proof of (ii) by induction.

4

## Bipartite subgraphs

**Theorem.** Every loopless multigraph  $G$  has a bipartite subgraph with at least  $e(G)/2$  edges.

*Proof # 1. Algorithmic.* (Start from an arbitrary bipartition and move over a vertex whose degree in its own part is *more* than its degree in the other part. Iterate. Prove termination. Prove that at termination you have what you want.)

*Proof # 2. Extremality.* (Consider a bipartite subgraph  $H$  with the *maximum number of edges*, prove that  $d_H(v) \geq d_G(v)/2$  for every vertex  $v \in V(G)$  and use the Handshaking Lemma.)

**Remark 1.** *Maximum vs. maximal.* Algorithmic proof *not* necessarily ends up in bipartite subgraph with maximum number of edges.

**Remark 2.** The constant multiplier  $\frac{1}{2}$  of  $e(G)$  in the Theorem is **best possible**. *Example:*  $K_n$ .