

Connectivity

A **separating set** (or **vertex cut**) of a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more than one component. For $G \neq K_n$, the **connectivity** of G is $\kappa(G) := \min\{|S| : S \text{ is a vertex cut}\}$. By definition, $\kappa(K_n) := n - 1$. A graph G is **k -connected** if there is no vertex cut of size $k - 1$. (i.e. $\kappa(G) \geq k$)

Examples. $\kappa(K_{n,m}) = \min\{n, m\}$
 $\kappa(Q_d) = d$

Extremal problem: What is the minimum number of edges in a k -connected graph?

Theorem. For every n , the minimum number of edges in a k -connected graph is $\lceil kn/2 \rceil$.

Proof:

$$\min \geq \lceil kn/2 \rceil, \text{ since } k \leq \kappa(G) \leq \delta(G)$$

$$\min \leq \lceil kn/2 \rceil; \text{ Example: Harary graphs } H_{k,n}.$$

Edge-connectivity

An **edge cut** of a multigraph G is an edge-set of the form $[S, \bar{S}]$, with $\emptyset \neq S \neq V(G)$ and $\bar{S} = V(G) \setminus S$.

For $S, T \subseteq V(G)$, $[S, T] := \{xy \in E(G) : x \in S, y \in T\}$.

The **edge-connectivity** of G is

$$\kappa'(G) := \min\{ |[S, \bar{S}]| : [S, \bar{S}] \text{ is an edge cut} \}.$$

A graph G is **k -edge-connected** if there is no edge cut of size $k - 1$ (i.e. $\kappa'(G) \geq k$).

Theorem. (Whitney, 1932) If G is a simple graph, then $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

Homework. Example of a graph G with $\kappa(G) = k$, $\kappa'(G) = l$, $\delta(G) = m$, for any $0 < k \leq l \leq m$.

Theorem. G is 3-regular $\Rightarrow \kappa(G) = \kappa'(G)$.

Characterization of 2-connected graphs_____

Theorem. (Whitney, 1932) Let G be a graph, $n(G) \geq 3$. Then G is 2-connected iff for every $u, v \in V(G)$ there exist two internally disjoint u, v -paths in G .

Theorem. Let G be a graph with $n(G) \geq 3$. Then the following four statements are equivalent.

- (i) G is 2-connected
- (ii) For all $x, y \in V(G)$, there are two internally disjoint x, y -paths.
- (iii) For all $x, y \in V(G)$, there is a cycle through x and y .
- (iv) $\delta(G) \geq 1$, and every pair of edges of G lies on a common cycle.

Expansion Lemma. Let G' be a supergraph of a k -connected graph G obtained by adding one vertex to $V(G)$ with at least k neighbors.

Then G' is k -connected as well.

Menger's Theorem

Given $x, y \in V(G)$, a set $S \subseteq V(G) \setminus \{x, y\}$ is an x, y -separator (or an x, y -cut) if $G - S$ has no x, y -path.

A set \mathcal{P} of paths is called **pairwise internally disjoint (p.i.d.)** if for any two path $P_1, P_2 \in \mathcal{P}$, P_1 and P_2 have no common internal vertices.

Define

$\kappa(x, y) := \min\{|S| : S \text{ is an } x, y\text{-cut,}\}$ and

$\lambda(x, y) := \max\{|\mathcal{P}| : \mathcal{P} \text{ is a set of p.i.d. } x, y\text{-paths}\}$

Local Vertex-Menger Theorem (Menger, 1927) Let $x, y \in V(G)$, such that $xy \notin E(G)$. Then

$$\kappa(x, y) = \lambda(x, y).$$

Corollary (Global Vertex-Menger Theorem) A graph G is k -connected iff for any two vertices $x, y \in V(G)$ there exist k p.i.d. x, y -paths.

Proof: Lemma. For every $e \in E(G)$, $\kappa(G - e) \geq \kappa(G) - 1$.

Edge-Menger

Given $x, y \in V(G)$, a set $F \subseteq E(G)$ is an x, y -**disconnecting set** if $G - F$ has no x, y -path. Define

$$\kappa'(x, y) := \min\{|F| : F \text{ is an } x, y\text{-disconnecting set,}\}$$

$$\lambda'(x, y) := \max\{|\mathcal{P}| : \mathcal{P} \text{ is a set of p.e.d.* } x, y\text{-paths}\}$$

* p.e.d. means **pairwise edge-disjoint**

Local Edge-Menger Theorem For all $x, y \in V(G)$,

$$\kappa'(x, y) = \lambda'(x, y).$$

Proof. Apply Menger's Theorem for the line graph of G' , where $V(G') = V(G) \cup \{s, t\}$ and $E(G') = E(G) \cup \{sx, yt\}$.

The **line graph** $L(G)$ of a graph G is defined by

$$V(L(G)) := E(G),$$

$$E(L(G)) := \{ef : e \text{ and } f \text{ share an endpoint}\}.$$

Corollary (Global Edge-Menger Theorem) Multigraph G is **k -edge-connected iff** there is a set of **k p.e.d. x, y -paths** for any two vertices x and y .

Directed graphs

A **directed (multi)graph** (or **digraph**) is a triple consisting of a vertex set $V(G)$, edge set $E(G)$, and a function assigning each edge an ordered pair of vertices.

For an edge $e = (x, y)$, x is the **tail** of e , y is its **head**.

By **path** and **cycle** in a **directed graph** we always mean directed path and directed cycle.

A directed graph is **weakly connected** if the underlying undirected graph is connected; it is **strongly connected** or **strong** if there is a u, v -path for any vertex u and any vertex $v \neq u$.

The **out-neighborhood** of v in G is

$$N_G^+(v) = \{w \in V(G) : (v, w) \in E(G)\}.$$

The **out-degree** of v is $d_G^+(v) = |N_G^+(v)|$.

The **in-neighborhood** of v in G is

$$N_G^-(v) = \{w \in V(G) : (w, v) \in E(G)\}.$$

The **in-degree** of v is $d_G^-(v) = |N_G^-(v)|$.

Déjà vu_____

Directed Handshaking. In a directed multigraph G , we have

$$\sum_{v \in V(G)} d^+(v) = e(G) = \sum_{v \in V(G)} d^-(v).$$

A directed multigraph is **Eulerian** if it has a directed Eulerian circuit, i.e. a closed directed trail containing all edges.

Theorem. A weakly connected directed multigraph on $n(D) \geq 2$ vertices is Eulerian **iff** $d^+(v) = d^-(v)$ for each vertex v .

Proof. Similar to the undirected case. Think it over.

Factors

A **factor** of a graph is a spanning subgraph. A **k -factor** is a spanning k -regular subgraph.

Every regular bipartite graph has a 1-factor.

Not every regular graph has a 1-factor.

But...

Theorem. (Petersen, 1891) Every $2k$ -regular graph has a 2-factor.

Proof. Use an Eulerian cycle of G to orient the edges of G and thus create a digraph D on $V(G)$ such that $d^+(v) = d^-(v) = k$ for every $v \in V(D)$.

Then from D define a k -regular bipartite graph H by

$$V(H) = \{x^+, x^- : x \in V(D)\} \text{ and}$$

$$E(H) = \{x^+y^- : (x, y) \in E(D)\}.$$

A perfect matching in H corresponds to a 2-factor in G .