Connectivity

A separating set (or vertex cut) of a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more than one component. For $G \neq K_n$, the connectivity of G is $\kappa(G) := min\{|S| : S \text{ is a vertex cut}\}$. By definition, $\kappa(K_n) := n - 1$. A graph G is k-connected if there is no vertex cut of size $k - 1$. (i.e. $\kappa(G) \geq k$)

Examples. $\kappa(K_{n,m}) = \min\{n, m\}$
$\kappa(Q_d) = d$

Extremal problem: What is the minimum number of edges in a k-connected graph?

Theorem. For every n, the minimum number of edges in a k-connected graph is $\lceil kn/2 \rceil$.

Proof:

min $\geq \lceil kn/2 \rceil$, since $k \leq \kappa(G) \leq \delta(G)$
min $\leq \lceil kn/2 \rceil$; Example: Harary graphs $H_{k,n}$.

Edge-connectivity

An edge cut of a multigraph G is an edge-set of the form $[S, \bar{S}]$, with $\emptyset \neq S \neq V(G)$ and $\bar{S} = V(G) \setminus S$.

For $S, T \subseteq V(G)$, $[S, T] := \{xy \in E(G) : x \in S, y \in T\}$.

The edge-connectivity of G is

$$\kappa'(G) := \min\{\ |[S, \bar{S}]|\ :\ [S, \bar{S}] \text{ is an edge cut}\}.$$

A graph G is k-edge-connected if there is no edge cut of size $k - 1$ (i.e. $\kappa'(G) \geq k$).

Theorem. (Whitney, 1932) If G is a simple graph, then $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

Homework. Example of a graph G with $\kappa(G) = k$, $\kappa'(G) = l$, $\delta(G) = m$, for any $0 < k \leq l \leq m$.

Theorem. G is 3-regular $\Rightarrow$ $\kappa(G) = \kappa'(G)$.

Characterization of 2-connected graphs

Theorem. (Whitney,1932) Let G be a graph, $n(G) \geq 3$. Then G is 2-connected iff for every $u, v \in V(G)$ there exist two internally disjoint u, v-paths in G.

Theorem. Let G be a graph with $n(G) \geq 3$. Then the following four statements are equivalent.

(i) G is 2-connected

(ii) For all $x, y \in V(G)$, there are two internally disjoint x, y-path.

(iii) For all $x, y \in V(G)$, there is a cycle through x and y.

(iv) $\delta(G) \geq 1$, and every pair of edges of G lies on a common cycle.

Expansion Lemma. Let G' be a supergraph of a k-connected graph G obtained by adding one vertex to $V(G)$ with at least k neighbors.
Then G' is k-connected as well.

Menger's Theorem

Given $x, y \in V(G)$, a set $S \subseteq V(G) \setminus \{x, y\}$ is an x, y-separator (or an x, y-cut) if $G - S$ has no x, y-path.
A set $\mathcal{P}$ of paths is called pairwise internally disjoint (p.i.d.) if for any two path $P_1, P_2 \in \mathcal{P}$, P_1 and P_2 have no common internal vertices.
Define

$$\kappa(x, y) := \min\{|S| : S \text{ is an } x, y\text{-cut,}\} \text{ and}$$
$$\lambda(x, y) := \max\{|\mathcal{P}| : \mathcal{P} \text{ is a set of p.i.d. } x, y\text{-paths}\}$$

Local Vertex-Menger Theorem (Menger, 1927) Let $x, y \in V(G)$, such that $xy \notin E(G)$. Then

$$\kappa(x, y) = \lambda(x, y).$$

Corollary (Global Vertex-Menger Theorem) A graph G is k-connected iff for any two vertices $x, y \in V(G)$ there exist k p.i.d. x, y-paths.

Proof: **Lemma.** For every $e \in E(G)$, $\kappa(G - e) \geq \kappa(G) - 1$.

Edge-Menger

Given $x, y \in V(G)$, a set $F \subseteq E(G)$ is an x, y-disconnecting set if $G - F$ has no x, y-path. Define

$\kappa'(x, y)$:= min{$|F|$: F is an x, y-disconnecting set,}
$\lambda'(x, y)$:= max{$|\mathcal{P}|$: $\mathcal{P}$ is a set of p.e.d.* x, y-paths}

* p.e.d. means pairwise edge-disjoint

Local Edge-Menger Theorem For all $x, y \in V(G)$,

$$\kappa'(x, y) = \lambda'(x, y).$$

Proof. Apply Menger's Theorem for the line graph of G', where $V(G') = V(G) \cup \{s, t\}$ and $E(G') = E(G) \cup \{sx, yt\}$.

The line graph $L(G)$ of a graph G is defi ned by
$V(L(G)) := E(G)$,
$E(L(G)) := \{ef$: e and f share an endpoint$\}$.

Corollary (Global Edge-Menger Theorem) Multigraph G is k-edge-connected iff there is a set of k p.e.d.x, y-paths for any two vertices x and y.

Directed graphs

A directed (multi)graph (or digraph) is a triple consisting of a vertex set $V(G)$, edge set $E(G)$, and a function assigning each edge an ordered pair of vertices.

For an edge $e = (x, y)$, x is the tail of e, y is its head.

By path and cycle in a directed graph we always mean directed path and directed cycle.

A directed graph is weakly connected if the underlying undirected graph is connected; it is strongly connected or strong if there is a u, v-path for any vertex u and any vertex $v \neq u$.

The out-neighborhood of v in G is
$N^+_G(v) = \{w \in V(G) : (v, w) \in E(G)\}$.
The out-degree of v is $d^+_G(v) = |N^+_G(v)|$.

The in-neighborhood of v in G is
$N^-_G(v) = \{w \in V(G) : (w, v) \in E(G)\}$.
The in-degree of v is $d^-_G(v) = |N^-_G(v)|$.

Déjà vu

Directed Handshaking. In a directed multigraph G, we have

$$\sum_{v \in V(G)} d^+(v) = e(G) = \sum_{v \in V(G)} d^-(v).$$

A directed multigraph is Eulerian if it has a directed Eulerian circuit, i.e. a closed directed trail containing all edges.

Theorem. A weakly connected directed multigraph on $n(D) \geq 2$ vertices is Eulerian iff $d^+(v) = d^-(v)$ for each vertex v.

Proof. Similar to the undirected case. Think it over.

Factors

A factor of a graph is a spanning subgraph. A k-factor is a spanning k-regular subgraph.

Every regular bipartite graph has a 1-factor.

Not every regular graph has a 1-factor.

But...

Theorem. (Petersen, 1891) Every $2k$-regular graph has a 2-factor.

Proof. Use an Eulerian cycle of G to orient the edges of G and thus create a digraph D on $V(G)$ such that $d^+(v) = d^-(v) = k$ for every $v \in V(D)$.
Then from D define a k-regular bipartite graph H by

$V(H) = \{x^+, x^- : x \in V(D)\}$ and
$E(H) = \{x^+y^- : (x, y) \in E(D)\}$.

A perfect matching in H corresponds to a 2-factor in G.