Menger's Theorem for directed graphs

Given $x, y \in V(D)$, a set $S \subseteq V(D) \setminus \{x, y\}$ is an x, y-separator (or an x, y-cut) if $D - S$ has no x, y-path.
Define

$$\kappa_D(x, y) := \min\{|S| : S \text{ is an } x, y\text{-cut,}\} \text{ and}$$
$$\lambda_D(x, y) := \max\{|\mathcal{P}| : \mathcal{P} \text{ is a set of p.i.d. } x, y\text{-paths}\}$$

Directed-Local-Vertex-Menger Theorem Let $x, y \in V(D)$, such that $\vec{xy} \notin E(D)$. Then

$$\kappa_D(x, y) = \lambda_D(x, y).$$

Proof. (Aharoni) Let $A = N^+(x)$ and $B = N^-(y)$.

$$\begin{aligned} D' := D - \{x, y\} \;&-\; \{\vec{za} : a \in A, z \in V(D)\} \\ &-\; \{\vec{bz} : b \in B, z \in V(D)\} \end{aligned}$$

$\mathcal{D}$: family of all A, B-paths in D'.

GOAL: Find a family $\mathcal{P} \subseteq \mathcal{D}$ of pairwise disjoint A, B-paths and a subset $S \subseteq V(D')$ such that
$|S \cap V(P)| \geq 1$ for every $P \in \mathcal{D}$ and
$|S \cap V(P)| = 1$ for every $P \in \mathcal{P}$.

Proving the GOAL is indeed enough. (Think it over)

Proof of GOAL. Define an auxiliary bipartite graph H.

$$\begin{aligned} V(H) &:= \{v^-, v^+ : v \in V(D')\} \\ E(H) &:= \{u^+v^- : uv \in E(D')\} \cup \\ &\quad\;\; \{v^-v^+ : v \in V(D') \setminus A \setminus B\} \end{aligned}$$

By König's Theorem there is a matching M and a vertex-cover C in H, such that $|e \cap C| = 1$ for every $e \in M$.

$$\begin{aligned} \mathcal{P} &:= \{x_1 \cdots x_k \in \mathcal{D} : x_i^+ x_{i+1}^- \in M \text{ for } 1 \leq i < k\}. \\ S &:= \{v \in V(D') \;:\; v^+, v^- \in C \text{ or } v^+ \in A^+ \cap C \\ &\qquad\qquad\qquad\quad \text{or } v^- \in B^- \cap C\}. \end{aligned}$$

- Any two paths $P_1, P_2 \in \mathcal{P}$ are disjoint.

  $V(P_1) \cap V(P_2) \neq \emptyset$ implies there is $f_1 \in E(P_1)$, $f_2 \in E(P_2)$ such that $f_1 \neq f_2$ and $f_1 \cap f_2 \neq \emptyset$. $P_1, P_2 \in \mathcal{P}$ implies that for any $f_i \in E(P_i)$ either $f_1 = f_2$ or $f_1 \cap f_2 = \emptyset$.
- Any A, B-path $x_0 x_1 x_2 \cdots x_k$ contains a vertex from S.

  Let i be the largest index such that $x_i^- \in C$. (There is such, unless $x_0^+ \in C$ and $i < k$ unless $x_k^- \in C$)
  Then $x_i^+ \in C$ since $x_i^+ x_{i+1}^-$ must be covered.
- No A, B-path $u_0 u_1 u_2 \cdots u_k = P \in \mathcal{P}$ contains more than one vertices from S.

  Suppose P does contain more. Let u_i and $u_j \in S \cap V(P)$ such that $u_k \notin S$ for $i < k < j$. Then $u_i^+, u_j^- \in C$ by definition of S. Let k be the largest index, $i < k < j$, such that $u_k^+ \in C$. Then $u_{k+1}^- \in C$ to cover the edge $u_{k+1}^- u_{k+1}^+$. Hence edge $u_k^+ u_{k+1}^- \in M$ is covered twice by C, a contradiction.

Corollaries

Corollary (Directed-Global-Vertex-Menger Theorem) A digraph D is strongly k-connected iff for any two vertices $x, y \in V(D)$ there exist k p.i.d. x, y-paths.

Proof: **Lemma.** For every $e \in E(D)$, $\kappa_D(G-e) \geq \kappa_D(G) - 1$.

The proof of the very first, the original Menger Theorem (the Undirected-Local-Vertex version) is

HOMEWORK !!!

Derive implication DLVM $\Rightarrow$ ULVM

Directed Edge-Menger

Given $x, y \in V(D)$, a set $F \subseteq E(D)$ is an x, y-disconnecting set if $D - F$ has no x, y-path. Define

$$\kappa'_D(x,y) := \text{min}\{|F| : F \text{ is an } x,y\text{-disconnecting set,}\}$$
$$\lambda'_D(x,y) := \text{max}\{|\mathcal{P}| : \mathcal{P} \text{ is a set of p.e.d.}^* \ x,y\text{-paths}\}$$

* p.e.d. means pairwise edge-disjoint

Directed-Local-Edge-Menger Theorem For all $x, y \in V(D)$,

$$\kappa'_D(x,y) = \lambda'_D(x,y).$$

Proof. Create directed line graph and apply DLVM.

Corollary (Directed-Global-Edge-Menger Theorem) Directed multigraph D is strongly k-edge-connected iff there is a set of k p.e.d.x, y-paths for any two vertices x and y.