

## Jordan Curves

A **curve** is a subset of  $\mathbb{R}^2$  of the form

$$\alpha = \{\gamma(x) : x \in [0, 1]\},$$

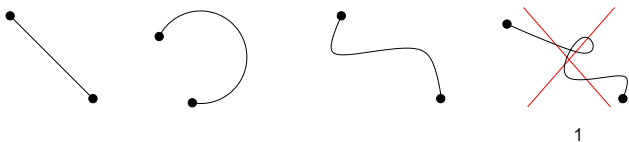
where  $\gamma : [0, 1] \rightarrow \mathbb{R}^2$  is a continuous mapping from the closed interval  $[0, 1]$  to the plane.  $\gamma(0)$  and  $\gamma(1)$  are called the *endpoints* of curve  $\alpha$ .

A curve is **closed** if its first and last points are the same. A curve is **simple** if it has no repeated points except possibly first = last. A closed simple curve is called a **Jordan-curve**.

Examples: Line segments between  $p, q \in \mathbb{R}^2$

$$x \mapsto xp + (1 - x)q,$$

circular arcs, Bezier-curves without self-intersection, etc...



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## Drawing of graphs

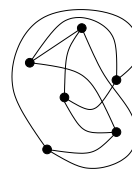
A **drawing** of a multigraph  $G$  is a function  $f$  defined on  $V(G) \cup E(G)$  that assigns

- a point  $f(v) \in \mathbb{R}^2$  to each vertex  $v$  and
- an  $f(u), f(v)$ -curve to each edge  $uv$ ,

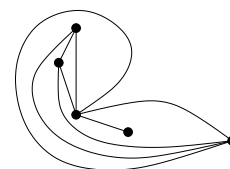
such that the images of vertices are distinct. A point in  $f(e) \cap f(e')$  that is not a common endpoint is a **crossing**.

A multigraph is **planar** if it has a drawing without crossings. Such a drawing is a **planar embedding** of  $G$ . A planar (multi)graph *together* with a particular planar embedding is called a **plane (multi)graph**.

drawing



plane embedding



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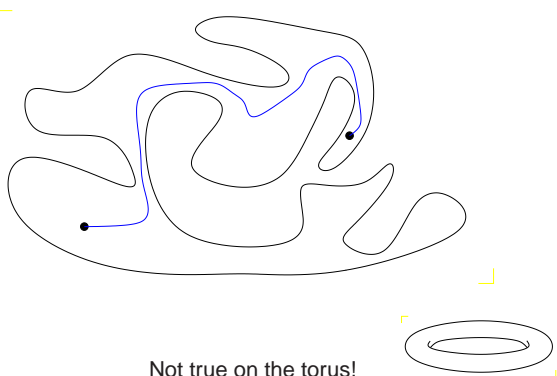
## Are there non-planar graphs?

**Proposition.**  $K_5$  and  $K_{3,3}$  cannot be drawn without crossing.

*Proof.* Define the *conflict graph* of edges.

### The unconscious ingredient.

**Jordan Curve Theorem.** A simple closed curve  $C$  partitions the plane into exactly two faces, each having  $C$  as boundary.



Not true on the torus!

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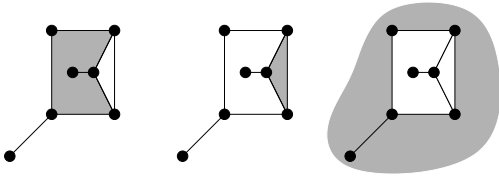
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3-a

## Regions and faces

An **open set** in the plane is a set  $U \subseteq \mathbb{R}^2$  such that for every  $p \in U$ , all points within some small distance belong to  $U$ . A **region** is an open set  $U$  that contains a  $u, v$ -curve for every pair  $u, v \in U$ . The **faces** of a plane multigraph are the maximal regions of the plane that contain no points used in the embedding.

A finite plane multigraph  $G$  has one **unbounded face** (also called **outer face**).



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## Dual graph

Denote the set of faces of a plane multigraph  $G$  by  $F(G)$  and let  $E(G) = \{e_1, \dots, e_m\}$ . Define the **dual** multigraph  $G^*$  of  $G$  by

- $V(G^*) := F(G)$
- $E(G^*) := \{e_1^*, \dots, e_m^*\}$ , where the endpoints of  $e_i^*$  are the two (not necessarily distinct) faces  $f', f'' \in F(G)$  on the two sides of  $e_i$ .

**Remarks.** Multiple edges and/or loops *could* appear in the dual of simple graphs

Different planar embeddings of the *same* planar graph could produce *different* duals.

**Proposition.** Let  $l(F_i)$  denote the length of face  $F_i$  in a plane multigraph  $G$ . Then

$$2e(G) = \sum l(F_i).$$

**Proposition.**  $e_1, \dots, e_r \in E(G)$  forms a cycle in  $G$  iff  $e_1^*, \dots, e_r^* \in E(G^*)$  forms a minimal nonempty edge-cut in  $G^*$ .

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## Euler's Formula

**Theorem.**(Euler, 1758) If a plane multigraph  $G$  with  $k$  components has  $n$  vertices,  $e$  edges, and  $f$  faces, then

$$n - e + f = 1 + k.$$

*Proof.* Induction on  $e$ .

**Base Case.** If  $e = 0$ , then  $n = k$  and  $f = 1$ .

Suppose now  $e > 0$ .

**Case 1.**  $G$  has a cycle.

Delete one edge from a cycle. In the new graph:

$e' = e - 1$ ,  $n' = n$ ,  $f' = f - 1$  (Jordan!), and  $k' = k$ .

**Case 2.**  $G$  is a forest.

Delete a pendant edge. In the new graph:

$e' = e - 1$ ,  $n' = n$ ,  $f' = f$ , and  $k' = k + 1$ .

**Remark.** The dual may depend on the embedding of the graph, but the number of faces does *not*.

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## Application – Platonic solids

- each face is congruent to the same regular convex  $r$ -gon,  $r \geq 3$
- the same number  $d$  of faces meet at each vertex,  $d \geq 3$

EXAMPLES: cube, tetrahedron

$$fr = 2e \quad vd = 2e$$

Substitute into Euler's Formula

$$\frac{2e}{d} - e + \frac{2e}{r} = 2$$

$$\frac{1}{d} + \frac{1}{r} = \frac{1}{2} + \frac{1}{e}$$

Crucial observation: either  $d$  or  $r$  is 3.

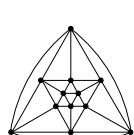
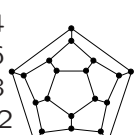
Possibilities:  $r \quad d \quad e \quad f \quad v$

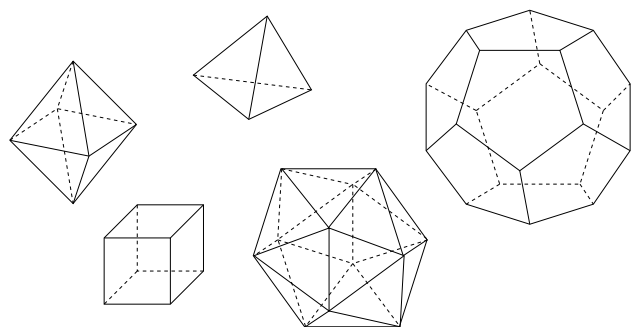
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## Applications of Euler's Formula \_\_\_\_\_

For a convex polytope,

$$\# \text{Vertices} - \# \text{Edges} + \# \text{Faces} = 2$$

	Tetrahedron	4	6	4	
	Cube	8	12	6	
	Octahedron	6	12	8	
	Dodecahedron	20	30	12	
	Icosahedron	12	30	20	



The platonic solids

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## Number of edges in a planar graphs \_\_\_\_\_

**Theorem.** If  $G$  is a simple, planar graph with  $n(G) \geq 3$ , then  $e(G) \leq 3n(G) - 6$ .

If also  $G$  is triangle-free, then  $e(G) \leq 2n(G) - 4$ .

*Proof.* Apply Euler's Formula.

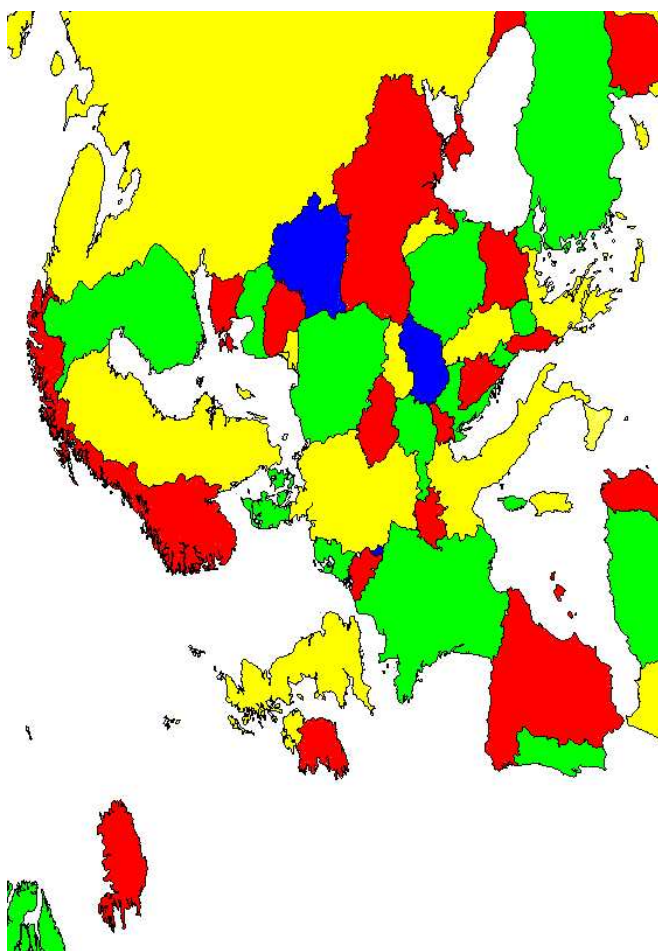
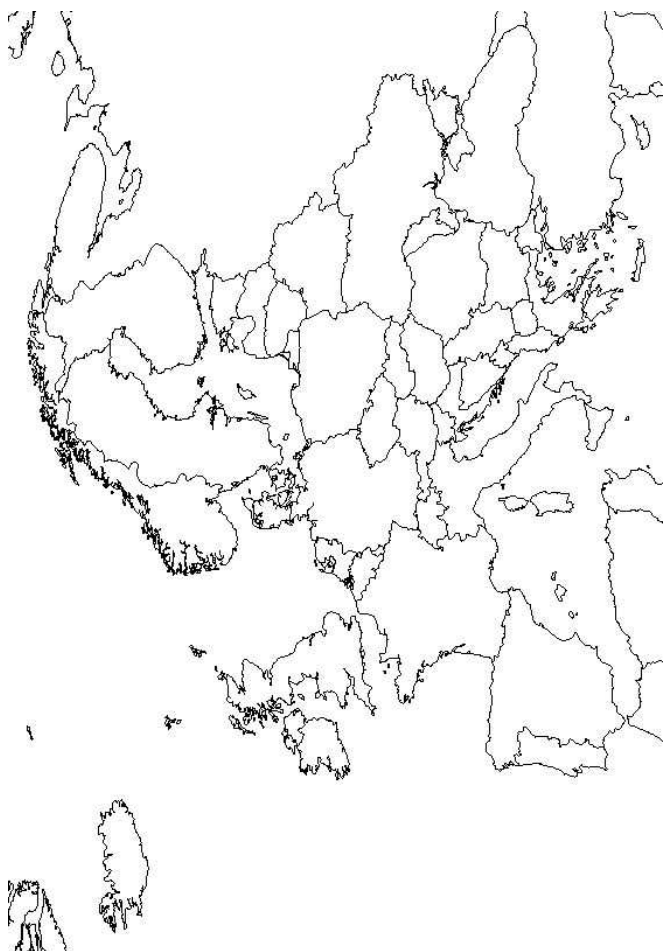
**Corollary**  $K_5$  and  $K_{3,3}$  are non-planar.

A **maximal planar graph** is a simple planar graph that is not a spanning subgraph of another planar graph. A **triangulation** is a simple plane graph where every face is a triangle.

**Proposition.** For a simple  $n$ -vertex plane graph  $G$ , the following are equivalent.

- A)  $G$  has  $3n - 6$  edges
- B)  $G$  is a triangulation.
- C)  $G$  is a maximal planar graph.

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## Coloring maps with 5 colors\_\_\_\_\_

**Five Color Theorem.** (Heawood, 1890) If  $G$  is planar, then  $\chi(G) \leq 5$ .

*Proof.* Take a minimal counterexample.

- (i) There is a vertex  $v$  of degree at most 5.
- (ii) Modify a proper 5-coloring of  $G - v$  to obtain a proper 5-coloring of  $G$ . A contradiction.

*Idea of modification:* Kempe chains.

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## Coloring maps with 4 colors\_\_\_\_\_

**Four Color Theorem.** (Appel-Haken, 1976) For any planar graph  $G$ ,  $\chi(G) \leq 4$ .

*Idea of the proof.*

W.l.o.g. we can assume  $G$  is a planar triangulation. A **configuration** in a planar triangulation is a separating cycle  $C$  (the **ring**) together with the portion of the graph inside  $C$ .

For the Four Color Problem, a set of configurations is an **unavoidable set** if a minimum counterexample must contain a member of it.

A configuration is **reducible** if a planar graph containing it cannot be a minimal counterexample.

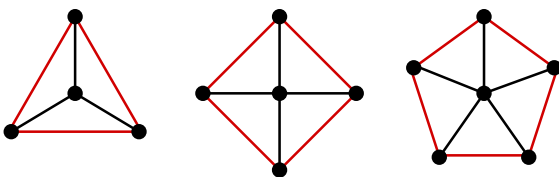
The usual proof attempts to

- (i) find a set  $\mathcal{C}$  of unavoidable configurations, and
- (ii) show that each configuration in  $\mathcal{C}$  is reducible.

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## Proof attempts of the Four Color Theorem\_\_\_\_

Kempe's original proof tried to show that the unavoidable set



is reducible.

Appel and Haken found an unavoidable set of 1936 configurations, (all with ring size at most 14) and proved each of them is reducible. (1000 hours of computer time)

Robertson, Sanders, Seymour and Thomas (1996) used an unavoidable set of 633 configurations. They used 32 rules to prove that each of them is reducible. (3 hours computer time)

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## Kuratowski's Theorem\_\_\_\_\_

**Theorem.**(Kuratowski, 1930) A graph  $G$  is planar iff  $G$  does not contain a subdivision of  $K_5$  or  $K_{3,3}$ .

*Outline of a proof.*

A **Kuratowski subgraph** of  $G$  is a subgraph of  $G$  that is a subdivision of  $K_5$  or  $K_{3,3}$ . A **minimal nonplanar graph** is a nonplanar graph such that every proper subgraph is planar.

Kuratowski's Theorem follows from the following Lemma and Theorem.

**Lemma** If  $G$  is a graph with fewest edges among counterexamples, then  $G$  is 3-connected.

**Lemma.** Every minimal nonplanar graph is 2-connected.

**Lemma.** Let  $S = \{x, y\}$  be a separating set of  $G$ . If  $G$  is a nonplanar graph, then adding the edge  $xy$  to some  $S$ -lobe of  $G$  yields a nonplanar graph.

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**Theorem.**(Tutte, 1960) If  $G$  is a 3-connected graph with no Kuratowski subgraph, then  $G$  has a convex embedding in the plane with no three vertices on a line.

A **convex embedding** of a graph is a planar embedding in which each face boundary is a convex polygon.

**Lemma.** If  $G$  is a 3-connected graph with  $n(G) \geq 5$ , then there is an edge  $e \in E(G)$  such that  $G \cdot e$  is 3-connected.

**Lemma.**  $G$  has no Kuratowski subgraph  $\Rightarrow G \cdot e$  has no Kuratowski subgraph.

## The Graph Minor Theorem\_\_\_\_\_

**Theorem.** (Robertson and Seymour, 1985-200?) In any infinite list of graphs, some graph is a minor of another.

*Proof:* more than 500 pages in 20 papers.

**Corollary** For any graph property that is closed under taking minors, there exists **finitely many** minimal **forbidden** minors.

*Homework.* Wagner's Theorem. Every nonplanar graph contains either a  $K_5$  or  $K_{3,3}$ -minor.

For embeddability on the **projective plane**, it is known that there are **35** minimal forbidden minors. For embeddability on the **torus**, we don't know the exact number of minimal forbidden minors; there are **more than 800 known**. (The generalization of Kuratowski's subdivision characterization yields an infinite list.)