Graph Theory — Old Exams

Exam — March 4th, 2003

1. (12 points) Let $d_1, \ldots, d_n$ be positive integers, with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, \ldots, d_n$ if and only if $\sum d_i = 2n-2$.

2. (12 points) Determine the maximum number of edges in a simple bipartite graph that contains no matching with 10 edges and no star with 8 edges.

3. (12 points) (a) Determine the number of triangles in the line graph of Q_5.

(b) Prove that the Petersen graph is non-planar.

4. (12 points) Prim's Algorithm grows a spanning tree from a given vertex of a connected weighted graph G, iteratively adding the cheapest edge from a vertex already reached to a vertex not yet reached, finishing when all the vertices of G have been reached. (Ties are broken arbitrarily.) Prove that Prim's algorithm produces a minimum-weight spanning tree of G.

5. (12 points) Prove that a tree T has a perfect matching if and only if $o(T-v) = 1$ for every $v \in V(T)$.

6. (18 points) (a) Prove that no matter how you color the edges of a K_{10} with RED/BLUE, you will find either a completely RED K_3 or a completely BLUE K_4. (Hint: You can apply the fact, that no matter how you color the edges of K_6 with RED/BLUE there will be a monochromatic triangle; We formulated and proved this statement in terms of acquaintances and non-acquaintances among six people.)

(b) Prove the same statement with K_9 instead of K_{10}! (Hint: Look at more carefully how a counterexample would look like.)

(c) Show that the same statement is not true for K_8.

Exam — March 2nd, 2004

1. (Not covered in this year's 2006/07 course!)

Find a maximum flow from s to t in the network below. Give a short proof that your flow is optimal. (The numbers on edges represent the capacities.)

2. Let $d_1, \ldots, d_n$ be positive integers, with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, \ldots, d_n$ if and only if $\sum_{i=1}^{n} d_i = 2n-2$.

3. Let G be a simple bipartite graph. Prove that the complement of the line graph of G is perfect. (A solution using the weak perfect graph theorem will receive only half of the credit.)

4. Let G be an $(r-1)$-edge-connected r-regular simple graph with even number of vertices, where r is a positive integer. Prove that G has a perfect matching.

5. Denote by $\mathcal{G}$ the class of bipartite graphs with 161 edges and maximum degree 7. Determine the minimum of the maximum matching size within the class $\mathcal{G}$, namely, $\min_{G\in\mathcal{G}} \alpha'(G)$.

6. Prove the Four Color Theorem for triangle-free planar graphs.

Exam — March 1st, 2005

1. Let T and T' be two spanning trees of a connected graph G. For any $e \in E(T) \setminus E(T')$, prove that there exists an edge $e' \in E(T') \setminus E(T)$, such that both $T' + e - e'$ and $T - e + e'$ are spanning trees of G.

2. Let G be a 3-regular graph. Prove that G has a vertex two-coloring, such that each monochromatic component has at most two vertices.

3. Somebody distributed a deck of cards into 13 packs of 4 cards. Show that you can select one of each kind by selecting one from each pack. (A deck contains 52 cards. Each card has a "suit" and a "kind". There are four suits: spades, heart, diamond, and clubs. There are 13 kinds: 2,3,4,5,6,7,8,9,10, J, Q, K, A.)

4. The Grötsch?? graph is created from C_5 by applying Myczeilski's?? construction. Construct a graph for every n which has no subgraph isomorphic to the Grötsch graph and contains $\lfloor \frac{n^2}{3} \rfloor$ edges.

5. Let G be a simple planar graph with at least four vertices. Prove that G has at least four vertices whose degree is less than 6.

6. Let $G \neq K_2$ be a connected, regular bipartite graph. Show that G is 2-connected.

Exam — March 7th, 2006

1. Prove or disprove: Every 4-regular planar graph has a triangle.

2. Prove that $X_l(G) + X_l(\bar{G}) \leq n + 1$.

3. Let G be an r-connected graph of even order having no $K_{1,r+1}$ as an induced subgraph. Prove that G has a perfect matching.

4. Let $k \leq n - 1$. Prove that if $\delta(G) \geq \frac{n+k-2}{2}$, then $\kappa(G) \geq k$.

5. Let G be a simple bipartite graph. Prove that the complement of the line graph of G is perfect. (A solution using the weak perfect graph theorem will receive only half of the credit.)

6. Let C be a cycle in a connected weighted graph. Let e be an edge of maximum weight on C. Prove that there is a minimum-weight spanning tree not containing e.
Use this to prove that iteratively deleting a heaviest non-cut-edge until the remaining graph is acyclic produces a minimum-weight spanning tree.