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Graph Theory — Problem Set 14

Course Webpage: `http://www.ti.inf.ethz.ch/ew/courses/GT06/`

Due date: None

Exercise 14.1

(Exercise 6.1.25 in the Textbook)

(!) Prove that every n-vertex plane multigraph isomorphic to its dual has $2n-2$ edges. For all $n \geq 4$, construct a simple n-vertex plane graph isomorphic to its dual.

Exercise 14.2

(Exercise 6.1.21 in the Textbook)

(!) Prove that a set of edges in a connected plane multigraph G forms a spanning tree of G if and only if the duals of the remaining edges form a spanning tree of G^*.

Exercise 14.3

(Exercise 6.1.35 in the Textbook)

(!) Prove that every simple planar graph with at least four vertices has at least four vertices with degree less than 6. For each even value of n with $n \geq 8$, construct an n-vertex simple planar graph G that has exactly four vertices with degree less than 6. (Grünbaum–Motzkin [1963])

Exercise 14.4

(Exercise not in the Textbook)

Prove that every outerplanar graph is 3-colorable following each of the ideas below.

a) Use Four Color Theorem.
b) Use Dirac's theorem.
c) Show every simple outerplanar graph is 2-degenerate.

Exercise 14.5

(Exercise not in the Textbook)

In a graph G, to **contract** an edge e with endpoints u, v is to replace u and v with a single vertex whose incident edges are the edges other than e that were incident to u or v. (Hence, the resulting graph has one less edge than G.) A graph H is a **minor** of G if a copy of H can be obtained from G by deleting and/or contracting edges of G, and/or by deleting vertices of G. (Note that H is supposed to be a simple graph; we also make the graph simple if there exist a loop or multiple edges.) In such a case we say that **G contains an H-minor**.

Show that the Petersen graph contains a K_5-minor and a $K_{3,3}$-minor.

Exercise 14.6

(Exercise 6.2.12 in the Textbook)

(!) Wagner [1937] proved that the following condition is necessary and sufficient for a graph G to be planar: G contains neither a K_5-minor nor $K_{3,3}$-minor.

a) Show that deletion and contraction of edges preserve planarity. Conclude from this that Wagner's condition is necessary.
b) Use Kuratowski's Theorem to prove that Wagner's condition is sufficient.

Kuratowski's Theorem [1930]:
A graph G is planar if and only if G does not contain a subdivision of K_5 or $K_{3,3}$.