## 7.5 Expected number of conflicts

Since every configuration involved in a conflict has been created in some step $r$ (we include step 0), the total number of conflicts is

$$\sum_{r=0}^{n} \sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} |K(\Delta)|,$$

where $\mathcal{T}_{-1} := \emptyset$. $\mathcal{T}_0$ consists of constantly many configurations only (namely those where the set of defining elements is the empty set), each of which is in conflict with at most all elements; moreover, no conflict is created in step $n$. Hence,

$$\sum_{r=0}^{n} \sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} |K(\Delta)| = O(n) + \sum_{r=1}^{n-1} \sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} |K(\Delta)|,$$

and we will bound the latter quantity. Let

$$K(r) := \sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} |K(\Delta)|, \quad r = 1, \ldots, n-1.$$

and $k(r) := E(K(r))$ the expected number of conflicts created in step $r$.

**Bounding** $k(r)$. We know that $\mathcal{T}_r$ arises from a random $r$-element set $R$. Fixing $R$, the backwards movie view tells us that $\mathcal{T}_{r-1}$ arises from $\mathcal{T}_r$ by deleting a random element $s$ of $R$. Thus,

$$\begin{aligned}
k(r) &= \frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{1}{r} \sum_{s \in R} \sum_{\Delta \in \mathcal{T}(R) \setminus \mathcal{T}(R \setminus \{s\})} |K(\Delta)| \\
&= \frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{1}{r} \sum_{s \in R} \sum_{\Delta \in \mathcal{T}(R), s \in D(\Delta)} |K(\Delta)| \\
&\leqslant \frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{d}{r} \sum_{\Delta \in \mathcal{T}(R)} |K(\Delta)|,
\end{aligned}$$

since in the sum over $s \in R$, every configuration is counted at most $d$ times. Since we can rewrite

$$\sum_{\Delta \in \mathcal{T}(R)} |K(\Delta)| = \sum_{y \in X \setminus R} |\{\Delta \in \mathcal{T}(R) : y \in K(\Delta)\}|,$$

we thus have

$$k(r) \leqslant \frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{d}{r} \sum_{y \in X \setminus R} |\{\Delta \in \mathcal{T}(R) : y \in K(\Delta)\}|.$$

To estimate this further, here is a simple but crucial

**Lemma 7.6** *The configurations in $\mathcal{T}(R)$ that are not in conflict with $y$ are the configurations in $\mathcal{T}(R \cup \{y\})$ that do not have $y$ in their defining set; in formulas:*

$$|\mathcal{T}(R)| - |\{\Delta \in \mathcal{T}(R) : y \in K(\Delta)\}| = |\mathcal{T}(R \cup \{y\})| - \deg(y, R \cup \{y\}).$$

The proof is a direct consequence of the definitions: every configuration in $\mathcal{T}(R)$ not in conflict with $y$ is by definition still present in $\mathcal{T}(R \cup \{y\})$ and still does not have $y$ in its defining set. And a configuration in $\mathcal{T}(R \cup \{y\})$ with $y$ not in its defining set is by definition already present in $\mathcal{T}(R)$ and already there not in conflict with $y$.

The lemma implies that

$$k(r) \leqslant k_1(r) - k_2(r) + k_3(r),$$

where

$$k_1(r) = \frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{d}{r} \sum_{y \in X \setminus R} |\mathcal{T}(R)|,$$

$$k_2(r) = \frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{d}{r} \sum_{y \in X \setminus R} |\mathcal{T}(R \cup \{y\})|,$$

$$k_3(r) = \frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{d}{r} \sum_{y \in X \setminus R} \deg(y, R \cup \{y\}).$$

**Estimating $k_1(r)$.** This is really simple.

$$\begin{aligned} k_1(r) &= \frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{d}{r} \sum_{y \in X \setminus R} |\mathcal{T}(R)| \\ &= \frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{d}{r}(n-r)|\mathcal{T}(R)| \\ &= \frac{d}{r}(n-r)t_r. \end{aligned}$$

**Estimating** $k_2(r)$. For this, we need to employ our earlier $(R, y) \mapsto (R \cup \{y\}, y)$ bijection again.

$$
\begin{aligned}
k_2(r) &= \frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{d}{r} \sum_{y \in X \setminus R} |\mathcal{T}(R \cup \{y\})| \\
&= \frac{1}{\binom{n}{r}} \sum_{Q \subseteq X, |Q|=r+1} \frac{d}{r} \sum_{y \in Q} |\mathcal{T}(Q)| \\
&= \frac{1}{\binom{n}{r+1}} \sum_{Q \subseteq X, |Q|=r+1} \frac{\binom{n}{r+1}}{\binom{n}{r}} \frac{d}{r}(r+1)|\mathcal{T}(Q)| \\
&= \frac{1}{\binom{n}{r+1}} \sum_{Q \subseteq X, |Q|=r+1} \frac{d}{r}(n-r)|\mathcal{T}(Q)| \\
&= \frac{d}{r}(n-r)t_{r+1} \\
&= \frac{d}{r+1}(n-(r+1))t_{r+1} + \frac{dn}{r(r+1)}t_{r+1} \\
&= k_1(r+1) + \frac{dn}{r(r+1)}t_{r+1}.
\end{aligned}
$$

**Estimating** $k_3(r)$. This is similar to $k_2(r)$ and in addition uses a fact that we have employed before: $\sum_{y \in Q} \deg(y, Q) \leqslant d|\mathcal{T}(Q)|$.

$$
\begin{aligned}
k_3(r) &= \frac{1}{\binom{n}{r}} \sum_{R \subseteq X, |R|=r} \frac{d}{r} \sum_{y \in X \setminus R} \deg(y, R \cup \{y\}) \\
&= \frac{1}{\binom{n}{r}} \sum_{Q \subseteq X, |Q|=r+1} \frac{d}{r} \sum_{y \in Q} \deg(y, Q) \\
&\leqslant \frac{1}{\binom{n}{r}} \sum_{Q \subseteq X, |Q|=r+1} \frac{d^2}{r}|\mathcal{T}(Q)| \\
&= \frac{1}{\binom{n}{r+1}} \sum_{Q \subseteq X, |Q|=r+1} \frac{\binom{n}{r+1}}{\binom{n}{r}} \frac{d^2}{r}|\mathcal{T}(Q)| \\
&= \frac{1}{\binom{n}{r+1}} \sum_{Q \subseteq X, |Q|=r+1} \frac{n-r}{r+1} \cdot \frac{d^2}{r}|\mathcal{T}(Q)| \\
&= \frac{d^2}{r(r+1)}(n-r)t_{r+1} \\
&= \frac{d^2 n}{r(r+1)}t_{r+1} - \frac{d^2}{r+1}t_{r+1}.
\end{aligned}
$$

**Summing up.** Let us recapitulate: the overall expected number of conflicts is $O(n)$ plus

$$\sum_{r=1}^{n-1} k(r) = \sum_{r=1}^{n-1} (k_1(r) - k_2(r) + k_3(r)).$$

Using our previous estimates, $k_1(2), \ldots, k_1(n-1)$ are canceled by the first terms of $k_2(1), \ldots, k_2(n-2)$. The second term of $k_2(r)$ can be combined with the first term of $k_3(r)$, so that we get

$$\begin{aligned}
\sum_{r=1}^{n-1} (k_1(r) - k_2(r) + k_3(r)) &\leqslant k_1(1) - \underbrace{k_1(n)}_{=0} + n \sum_{r=1}^{n-1} \frac{d(d-1)}{r(r+1)} t_{r+1} - \sum_{r=1}^{n-1} \frac{d^2}{r+1} t_{r+1} \\
&\leqslant d(n-1)t_1 + d(d-1)n \sum_{r=1}^{n-1} \frac{t_{r+1}}{r(r+1)} \\
&= O\left(d^2 n \sum_{r=1}^{n} \frac{t_r}{r^2}\right).
\end{aligned}$$

**The Delaunay case.** We have argued that the expected number of conflicts asymptotically bounds the expected total location cost over all insertion steps. The previous equation tells us that this cost is proportional to $O(n)$ plus

$$O\left(9n \sum_{r=1}^{n} \frac{2(r+4)-4}{r^2}\right) = O\left(n \sum_{r=1}^{n} \frac{1}{r}\right) = O(n \log n).$$

Here,

$$\sum_{r=1}^{n} \frac{1}{r} =: H_n$$

is the $n$-th Harmonic Number which is known to be approximately $\ln n$.

By going through the abstract framework of configuration spaces, we have thus analyzed the randomized incremental construction of the Delaunay triangulation of $n$ points. According to Section 7.3, the expected update cost itself is only $O(n)$. The steps dominating the runtime are the location steps via the history graph. According to Section 7.5, all history graph searches (whose number is proportional to the number of conflicts) can be performed in expected time $O(n \log n)$, and this then also bounds the space requirements of the algorithm.

**Exercise 7.7** *Design and analyze a sorting algorithm based on randomized incremental construction in configuration spaces. The input is a set $S$ of numbers, and the output should be the sorted sequence (in increasing order).*

a) *Define an appropriate configuration space for the problem! In particular, the set of active configurations w.r.t.* $S$ *should represent the desired sorted sequence.*

b) *Provide an efficient implementation of the incremental construction algorithm. "Efficient" means that the runtime of the algorithm is asymptotically dominated by the number of conflicts.*

c) *What is the expected number of conflicts (and thus the asymptotic runtime of your sorting algorithm) for a set* $S$ *of* $n$ *numbers?*

## Questions

28. *What is a configuration space?* Give a precise definition! What is an active configuration?

29. *How do we get a configuration space from the problem of computing the Delaunay triangulation of a finite point set?*

30. *How many new active configurations do we get on average when inserting the* $r$*-th element?* Provide an answer for configuration spaces in general, and for the special case of the Delaunay triangulation.

31. *What is a conflict?* Provide an answer for configuration spaces in general, and for the special case of the Delaunay triangulation.

32. *Explain why counting the expected number of conflicts asymptotically bounds the cost for the history searches during the randomized incremental construction of the Delaunay triangulation!*