Chapter 8

Voronoi Diagrams

8.1 Post Office Problem

Suppose there are n post offices $p_1, \ldots p_n$ in a city. Someone who is located at a position q within the city would like to know which post office is closest to him. Modeling the city as a planar region, we think of $p_1, \ldots p_n$ and q as points in the plane. Denote the set of post offices by $P = \{p_1, \ldots p_n\}$.

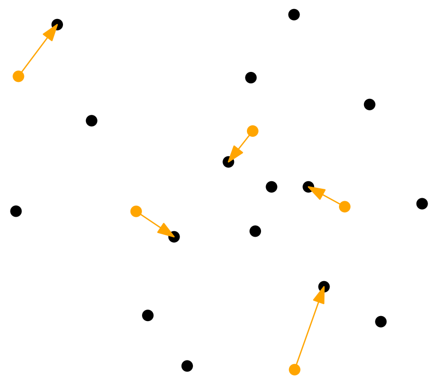


Figure 8.1: *Closest post offices for various query points.*

While the locations of post offices are known and do not change so frequently, we do not know in advance for which—possibly many—query locations the closest post office is to be found. Therefore, our long term goal is to come up with a data structure on top of P that allows to answer any possible query efficiently. The basic idea is to apply a so-called *locus approach*: we partition the query space into regions on which is the answer is the same. In our case, this amounts to partition the plane into regions such that for all points within a region the same point from P is closest (among all points from P).

As a warmup, consider the problem for two post offices $p_i, p_j \in P$. For which query locations is the answer p_i rather than p_j? This region is bounded by the bisector of p_i and p_j, that is, the set of points which have the same distance to both points.

Proposition 8.1 *For any two distinct points in* $\mathbb{R}^d$ *the bisector is a hyperplane, that is, in* $\mathbb{R}^2$ *it is a line.*

Proof. Let $p = (p_1, \ldots, p_d)$ and $q = (q_1, \ldots, q_d)$ be two points in $\mathbb{R}^d$. The bisector of p and q consists of those points $x = (x_1, \ldots, x_d)$ for which

$$\|p - x\| = \|q - x\| \iff \|p - x\|^2 = \|q - x\|^2 \iff \|p\|^2 - \|q\|^2 = 2(p - q)^\top x \,.$$

As p and q are distinct, this is the equation of a hyperplane. □

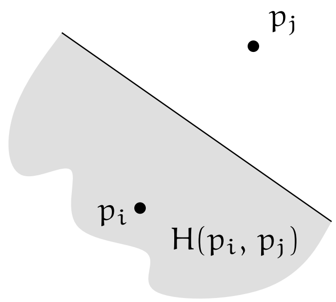


Figure 8.2: *The bisector of two points.*

Denote by $H(p_i, p_j)$ the closed halfspace bounded by the bisector of p_i and p_j that contains p_i. In $\mathbb{R}^2$, the region $H(p_i, p_j)$ is a halfplane; see Figure 8.2.

Exercise 8.2

a) What is the bisector of a line ℓ *and a point* $p \in \mathbb{R}^2 \setminus \ell$*, that is, the set of all points* $x \in \mathbb{R}^2$ *with* $\|x - p\| = \|x - \ell\|$ $(= \min_{q \in \ell} \|x - q\|)$*?*

b) For two points $p \neq q \in \mathbb{R}^2$*, what is the region that contains all points whose distance to* p *is exactly twice their distance to* q*?*

8.2 Voronoi Diagram

In the following we work with a set $P = \{p_1, \ldots, p_n\}$ of points in $\mathbb{R}^2$.

Definition 8.3 (Voronoi cell) *For* $p_i \in P$ *denote the* **Voronoi cell** $V_P(i)$ *of* p_i *by*

$$V_P(i) := \left\{ q \in \mathbb{R}^2 \mid \|q - p_i\| \leqslant \|q - p\| \text{ for all } p \in P \right\}.$$

Proposition 8.4

$$V_P(i) = \bigcap_{j \neq i} H(p_i, p_j) \,.$$

Proof. For $j \neq i$ we have $\|q - p_i\| \leqslant \|q - p_j\| \iff q \in H(p_i, p_j)$. □

Corollary 8.5 $V_P(i)$ *is non-empty and convex.*

Proof. According to Proposition 8.4, the region $V_P(i)$ is the intersection of a finite number of halfplanes and hence convex. As $p_i \in V_P(i)$, we have $V_P(i) \neq \emptyset$. □

Observe that every point of the plane lies in some Voronoi cell but no point lies in the interior of two Voronoi cells. Therefore these cells form a *subdivision* of the plane (a partition[1] into interior-disjoint simple polygons). See Figure 8.3 for an example.

Definition 8.6 (Voronoi Diagram) *The* **Voronoi Diagram** $VD(P)$ *of a set* $P = \{p_1, \ldots, p_n\}$ *of points in* $\mathbb{R}^2$ *is the subdivision of the plane induced by the Voronoi cells* $V_P(i)$*, for* $i = 1, \ldots, n$*. Denote by* $VV(P)$ *the set of vertices, by* $VE(P)$ *the set of edges, and by* $VR(P)$ *the set of regions (faces) of* $VD(P)$*.*

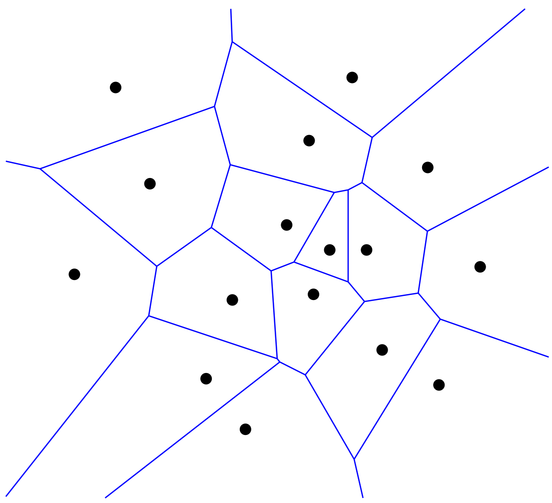


Figure 8.3: *Example: The Voronoi diagram of a point set.*

Lemma 8.7 *For every vertex* $v \in VV(P)$ *the following statements hold.*

a) v *is the common intersection of at least three edges from* $VE(P)$*;*

b) v *is incident to at least three regions from* $VR(P)$*;*

c) v *is the center of a circle* $C(v)$ *through at least three points from* P *such that*

d) $C(v)^\circ \cap P = \emptyset$*.*

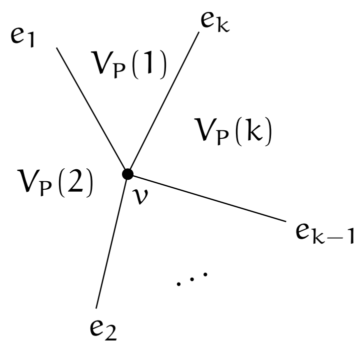
Proof. Consider a vertex $v \in VV(P)$. As all Voronoi cells are convex, $k \geqslant 3$ of them must be incident to v. This proves Part a) and b).

[1] Strictly speaking, to obtain a partition, we treat the shared boundaries of the polygons as separate entities.

Without loss of generality let these cells be $V_P(i)$, for $1 \leqslant i \leqslant k$. Denote by e_i, $1 \leqslant i \leqslant k$, the edge incident to v that bounds $V_P(i)$ and $V_P((i \bmod k) + 1)$.

For any $i = 1, \ldots, k$ we have $v \in e_i \Rightarrow \|v - p_i\| = \|v - p_{(i \bmod k)+1}\|$. In other words, p_1, p_2, …, p_k are cocircular, which proves Part c).

Part d): Suppose there exists a point $p_\ell \in C(v)^\circ$. Then the vertex v is closer to p_ℓ than it is to any of $p_1, \ldots, p_k$, in contradiction to the fact that v is contained in all of $V_P(1), \ldots, V_P(k)$. □



Corollary 8.8 *If* P *is in general position (no four points from* P *are cocircular), then for every vertex* $v \in VV(P)$ *the following statements hold.*

a) v *is the common intersection of exactly three edges from* $VE(P)$*;*

b) v *is incident to exactly three regions from* $VR(P)$*;*

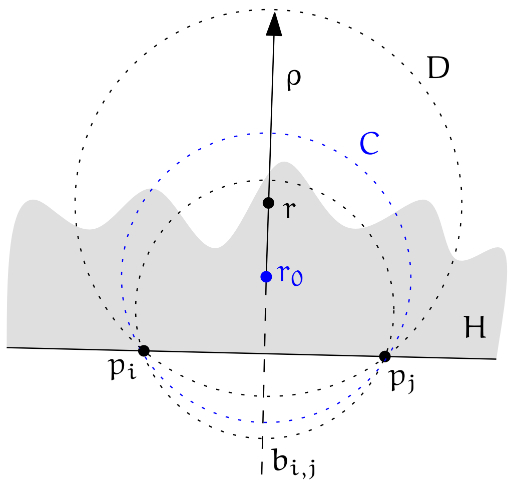
c) v *is the center of a circle* $C(v)$ *through exactly three points from* P *such that*

d) $C(v)^\circ \cap P = \emptyset$. □

Lemma 8.9 *There is an unbounded Voronoi edge bounding* $V_P(i)$ *and* $V_P(j)$ $\iff$ $\overline{p_i p_j} \cap P = \{p_i, p_j\}$ *and* $\overline{p_i p_j} \subseteq \partial\mathrm{conv}(P)$*, where the latter denotes the boundary of the convex hull of* P.

Proof.

Denote by $b_{i,j}$ the bisector of p_i and p_j, and let $\mathcal{D}$ denote the family of disks centered at some point on $b_{i,j}$ and passing through p_i (and p_j). There is an unbounded Voronoi edge bounding $V_P(i)$ and $V_P(j)$ $\iff$ there is a ray $\rho \subset b_{i,j}$ such that $\|r - p_k\| > \|r - p_i\|$ $(= \|r - p_j\|)$, for every $r \in \rho$ and every $p_k \in P$ with $k \notin \{i, j\}$. Equivalently, there is a ray $\rho \subset b_{i,j}$ such that for every point $r \in \rho$ the disk $C \in \mathcal{D}$ centered at r does not contain any point from P in its interior.



The latter statement implies that the open halfplane H, whose bounding line passes through p_i and p_j and such that H contains the infinite part of ρ, contains no point from P in its interior. Therefore, $\overline{p_i p_j}$ appears on $\partial\mathrm{conv}(P)$ and $\overline{p_i p_j}$ does not contain any $p_k \in P$, for $k \neq i, j$.

Conversely, suppose that $\overline{p_i p_j}$ appears on $\partial\mathrm{conv}(P)$ and $\overline{p_i p_j} \cap P = \{p_i, p_j\}$. Then some halfplane H whose bounding line passes through p_i and p_j contains no point from

P in its interior. In particular, the existence of H together with $\overline{p_i p_j} \cap P = \{p_i, p_j\}$ implies that there is some disk $C \in \mathcal{D}$ such that $C \cap P = \{p_i, p_j\}$. Denote by r_0 the center of C and let ρ denote the ray starting from r_0 along $b_{i,j}$ such that the infinite part of ρ is contained in H. Consider any disk $D \in \mathcal{D}$ centered at a point $r \in \rho$ and observe that $D \setminus H \subseteq C \setminus H$. As neither H nor C contain any point from P in their respective interior, neither does D. This holds for every D, and we have seen above that this statement is equivalent to the existence of an unbounded Voronoi edge bounding $V_P(i)$ and $V_P(j)$. □

8.3 Duality

A *straight-line dual* of a plane graph G is a graph G′ defined as follows: Choose a point for each face of G and connect any two such points by a straight edge, if the corresponding faces share an edge of G. Observe that this notion depends on the embedding; that is why the straight-line dual is defined for a plane graph rather than for an abstract graph. In general, G′ may have edge crossings, which may also depend on the choice of representative points within the faces. However, for Voronoi diagrams is a particularly natural choice of representative points such that G′ is plane: the points from P.

Theorem 8.10 (Delaunay [2]) *The straight-line dual of* $VD(P)$ *for a set* $P \subset \mathbb{R}^2$ *of* $n \geqslant 3$ *points in general position (no three points from* P *are collinear and no four points from* P *are cocircular) is a triangulation: the unique Delaunay triangulation of* P.

Proof. By Lemma 8.9, the convex hull edges appear in the straight-line dual T of VD(P) and they correspond exactly to the unbounded edges of VD(P). All remaining edges of VD(P) are bounded, that is, both endpoints are Voronoi vertices. Consider some $v \in VV(P)$. According to Corollary 8.8(b), v is incident to exactly three Voronoi regions, which, therefore, form a triangle $\triangle(v)$ in T. By Corollary 8.8(d), the circumcircle of $\triangle(v)$ does not contain any point from P in its interior. Hence $\triangle(v)$ appears in the (unique by Corollary 5.17) Delaunay triangulation of P.

Conversely, for any triangle $p_i p_j p_k$ in the Delaunay triangulation of P, by the empty circle property the circumcenter c of $p_i p_j p_k$ has p_i, p_j, and p_k as its closest points from P. Therefore, $c \in VV(P)$ and—as above—the triangle $p_i p_j p_k$ appears in T. □

It is not hard to generalize Theorem 8.10 to general point sets. In this case, a Voronoi vertex of degree k is mapped to a convex polygon with k cocircular vertices. Any triangulation of such a polygon yields a Delaunay triangulation of the point set.

Corollary 8.11 $|VE(P)| \leqslant 3n - 6$ *and* $|VV(P)| \leqslant 2n - 5$.

Proof. Every edge in VE(P) corresponds to an edge in the dual Delaunay triangulation. The latter is a plane graph on n vertices and thus has at most $3n - 6$ edges and at most $2n - 4$ faces by Corollary 2.5. Only the bounded faces correspond to a vertex in VD(P). □

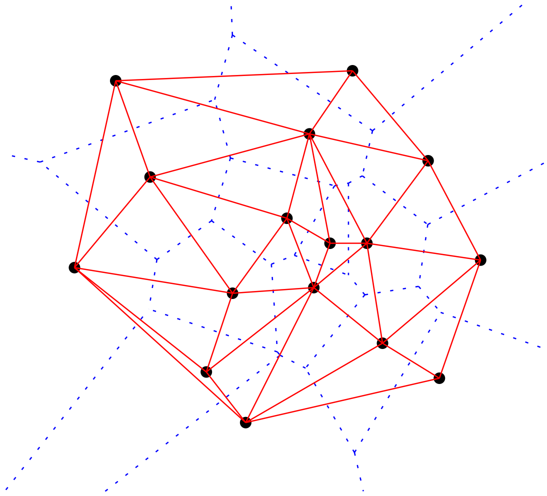


Figure 8.4: *The Voronoi diagram of a point set and its dual Delaunay triangulation.*

Corollary 8.12 *For a set* $P \subset \mathbb{R}^2$ *of* n *points, the Voronoi diagram of* P *can be constructed in expected* $O(n \log n)$ *time and* $O(n)$ *space.*

Proof. We have seen that a Delaunay triangulation T for P can be obtained using randomized incremental construction in the given time and space bounds. As T is a plane graph, its number of vertices, edges, and faces all are linear in n. Therefore, the straight-line dual of T—which by Theorem 8.10 is the desired Voronoi diagram—can be computed in $O(n)$ additional time and space. □

Exercise 8.13 *Consider the Delaunay triangulation* T *for a set* $P \subset \mathbb{R}^2$ *of* $n \geqslant 3$ *points in general position. Prove or disprove:*

a) Every edge of T *intersects its dual Voronoi edge.*

b) Every vertex of $\mathrm{VD}(P)$ *is contained in its dual Delaunay triangle.*

8.4 Lifting Map

Recall the lifting map that we used in Section 5.3 to prove that the Lawson Flip Algorithm terminates. Denote by $\mathcal{U} : z = x^2 + y^2$ the unit paraboloid in $\mathbb{R}^3$. The lifting map $\ell : \mathbb{R}^2 \to \mathcal{U}$ with $\ell : p = (p_x, p_y) \mapsto (p_x, p_y, {p_x}^2 + {p_y}^2)$ is the projection of the x/y-plane onto $\mathcal{U}$ in direction of the z-axis.

For $p \in \mathbb{R}^2$ let H_p denote the plane of tangency to $\mathcal{U}$ in $\ell(p)$. Denote by $h_p : \mathbb{R}^2 \to H_p$ the projection of the x/y-plane onto H_p in direction of the z-axis (see Figure 8.5).

Lemma 8.14 $||\ell(q) - h_p(q)|| = ||p - q||^2$, *for any points* $p, q \in \mathbb{R}^2$.

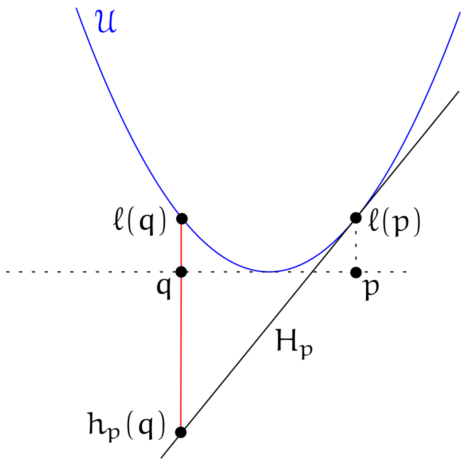


Figure 8.5: *Lifting map interpretation of the Voronoi diagram in a two-dimensional projection.*

Exercise 8.15 *Prove Lemma 8.14.* Hint: *First determine the equation of the tangent plane* H_p *to* $\mathcal{U}$ *in* $\ell(p)$.

Theorem 8.16 *For* $p = (p_x, p_y) \in \mathbb{R}^2$ *denote by* H_p *the plane of tangency to the unit paraboloid* $\mathcal{U} = \{(x, y, z) : z = x^2 + y^2\} \subset \mathbb{R}^3$ *in* $\ell(p) = (p_x, p_y, {p_x}^2 + {p_y}^2)$. *Let* $\mathcal{H}(P) := \bigcap_{p \in P} H_p^+$ *the intersection of all halfspaces above the planes* H_p, *for* $p \in P$. *Then the vertical projection of* $\partial\mathcal{H}(P)$ *onto the* x/y*-plane forms the Voronoi Diagram of* P *(the faces of* $\partial\mathcal{H}(P)$ *correspond to Voronoi regions, the edges to Voronoi edges, and the vertices to Voronoi vertices).*

Proof. For any point $q \in \mathbb{R}^2$, the vertical line through q intersects every plane H_p, $p \in P$. By Lemma 8.14 the topmost plane intersected belongs to the point from P that is closest to q. □

8.5 Point location in a Voronoi Diagram

One last bit is still missing in order to solve the post office problem optimally.

Theorem 8.17 *Given a triangulation* T *for a set* $P \subset \mathbb{R}^2$ *of* n *points, one can build in* $O(n)$ *time an* $O(n)$ *size data structure that allows for any query point* $q \in \mathrm{conv}(P)$ *to find in* $O(\log n)$ *time a triangle from* T *containing* q.

The data structure we will employ is known as *Kirkpatrick's hierarchy*. But before discussing it in detail, let us put things together in terms of the post office problem.

Corollary 8.18 (Nearest Neighbor Search) *Given a set* $P \subset \mathbb{R}^2$ *of* n *points, one can build in expected* $O(n \log n)$ *time an* $O(n)$ *size data structure that allows for any query* *point* $q \in \text{conv}(P)$ *to find in* $O(\log n)$ *time a nearest neighbor of* q *among the points from* P.

Proof. First construct the Voronoi Diagram V of P in expected $O(n \log n)$ time. It has exactly n convex faces. Every unbounded face can be cut by the convex hull boundary into a bounded and an unbounded part. As we are concerned with query points within $\text{conv}(P)$ only, we can restrict our attention to the bounded parts.[2] Any convex polygon can easily be triangulated in time linear in its number of edges (= number of vertices). As V has at most $3n - 6$ edges and every edge appears in exactly two faces, V can be triangulated in $O(n)$ time overall. Label each of the resulting triangles with the point from p, whose Voronoi region contains it, and apply the data structure from Theorem 8.17. □

8.5.1 Kirkpatrick's Hierarchy

We will now the develop the data structure for point location in a triangulation, as described in Theorem 8.17. For simplicity we assume that the triangulation T we work with is a maximal planar graph, that is, the outer face is a triangle as well. This can easily be achieved by an initial normalization step that puts a huge triangle T_h around T and triangulates the region in between T_h and T (in linear time).

The main idea for the data structure is to construct a hierarchy $T_0, \ldots, T_h$ of triangulations, such that

- $T_0 = T$,
- the vertices of T_i are a subset of the vertices of T_{i-1}, for $i = 1, \ldots, h$, and
- T_h is a single triangle only.

Search. For a query point x the triangle from T containing x can be found as follows.

Search$(x \in \mathbb{R}^2)$

1. For $i = h, h-1, \ldots, 0$: Find a triangle t_i from T_i that contains x.
2. return t_0.

This search is efficient under the following conditions.

(C1) Every triangle from T_i intersects only few ($\leqslant c$) triangles from T_{i-1}. (These will then be connected via the data structure.)

(C2) h is small ($\leqslant d \log n$).

[2] We even know how to decide in $O(\log n)$ time whether or not a given point lies within $\text{conv}(P)$, see Exercise 4.22.