Chapter 9

Line Arrangements

During the course of this lecture we encountered several situations where it was convenient to assume that a point set is "in general position". In the plane, general position usually amounts to no three points being collinear and/or no four of them being cocircular. This raises an algorithmic question: How can we test for n given points whether or not three of them are collinear? Obviously, we can test all triples in $O(n^3)$ time. Can we do better? Yes, we can! Using a detour through the so-called dual plane, we will see that this problem can be solved in $O(n^2)$ time. However, the exact algorithmic complexity of this innocent-looking problem is not known. In fact, to determine this complexity is one of the major open problems in theoretical computer science.

We will get back to the complexity theoretic problems and ramifications at the end of this chapter. But first let us discuss how to obtain a quadratic time algorithm to test whether n given points in the plane are in general position. This algorithm is a nice application of the projective duality transform, as defined below. Such transformations are very useful because they allow us to gain a new perspective on a problem by formulating it in a different but equivalent form. Sometimes such a *dual* form of the problem is easier to work with and—given that it is equivalent to the original *primal* form—any solution to the dual problem can be translated back into a solution to the primal problem.

So what is this duality transform about? Observe that points and hyperplanes in $\mathbb{R}^d$ are very similar objects, given that both can be described using d coordinates/parameters. It is thus tempting to match these parameters to each other and so create a mapping between points and hyperplanes. In $\mathbb{R}^2$ hyperplanes are lines and the standard *projective duality* transform maps a point $p = (p_x, p_y)$ to the line $p^* : y = p_x x - p_y$ and a non-vertical line $g : y = mx + b$ to the point $g^* = (m, -b)$.

Proposition 9.1 *The standard projective duality transform is*

- *incidence preserving:* $p \in g \iff g^* \in p^*$ *and*
- *order preserving:* p *is above* $g \iff g^*$ *is above* p^*.

Exercise 9.2 *Prove Proposition 9.1.*

Exercise 9.3 *Describe the image of the following point sets under this mapping*

a) a halfplane

b) $k \geqslant 3$ *collinear points*

c) a line segment

d) the boundary points of the upper convex hull of a finite point set.

Another way to think of duality is in terms of the parabola $\mathcal{P} : y = \frac{1}{2}x^2$. For a point p on $\mathcal{P}$, the dual line p^* is the tangent to $\mathcal{P}$ at p. For a point p not on $\mathcal{P}$, consider the vertical projection p' of p onto $\mathcal{P}$: the slopes of p^* and p'^* are the same, just p^* is shifted by the difference in y-coordinates.

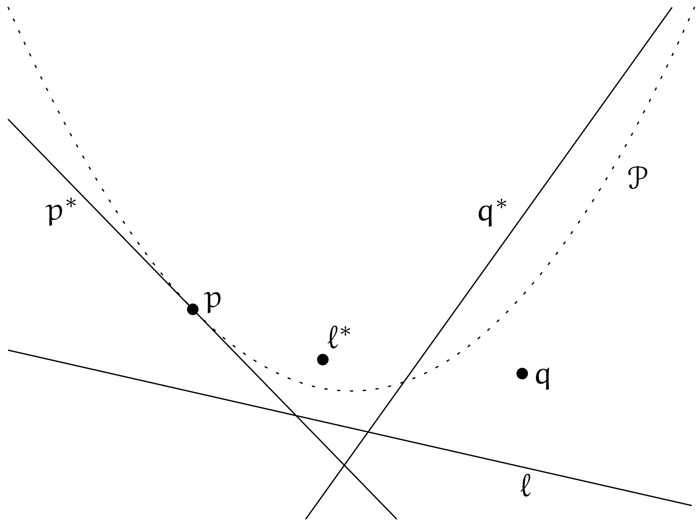


Figure 9.1: *Point* $\leftrightarrow$ *line duality with respect to the parabola* $\mathcal{P} : y = \frac{1}{2}x^2$.

The question of whether or not three points in the primal plane are collinear transforms to whether or not three lines in the dual plane meet in a point. This question in turn we will answer with the help of *line arrangements*, as defined below.

9.1 Arrangements

The subdivision of the plane induced by a finite set L of lines is called the **arrangement** $\mathcal{A}(L)$. We may imagine the creation of this subdivision as a recursive process, defined by the given set L of lines. As a first step, remove all lines (considered as point sets) from the plane $\mathbb{R}^2$. What remains of $\mathbb{R}^2$ are a number of open connected components (possibly only one), which we call the (2-dimensional) **cells** of the subdivision. In the next step, from every line in L remove all the remaining lines (considered as point sets). In this way every line is split into a number of open connected components (possibly only one), which collectively form the (1-dimensional cells or) **edges** of the subdivision. What remains of the lines are the (0-dimensional cells or) **vertices** of the subdivision, which are intersection points of lines from L.

Observe that all cells of the subdivision are intersections of halfplanes and thus convex. A line arrangement is **simple** if no two lines are parallel and no three lines meet in a point. Although lines are unbounded, we can regard a line arrangement a bounded object by (conceptually) putting a sufficiently large box around that contains all vertices. Such a box can be constructed in $O(n \log n)$ time for n lines.

Exercise 9.4 *How?*

Moreover, we can view a line arrangement as a planar graph by adding an additional vertex at "infinity", that is incident to all rays which leave this bounding box. For algorithmic purposes, we will mostly think of an arrangement as being represented by a doubly connected edge list (DCEL), cf. Section 2.2.1.

Theorem 9.5 *A simple arrangement $\mathcal{A}(L)$ of n lines in $\mathbb{R}^2$ has $\binom{n}{2}$ vertices, n^2 edges, and $\binom{n}{2} + n + 1$ faces/cells.*

Proof. Since all lines intersect and all intersection points are pairwise distinct, there are $\binom{n}{2}$ vertices.

The number of edges we count using induction on n. For $n = 1$ we have $1^2 = 1$ edge. By adding one line to an arrangement of $n-1$ lines we split $n-1$ existing edges into two and introduce n new edges along the newly inserted line. Thus, there are in total $(n-1)^2 + 2n - 1 = n^2 - 2n + 1 + 2n - 1 = n^2$ edges.

The number f of faces can now be obtained from Euler's formula $v - e + f = 2$, where v and e denote the number of vertices and edges, respectively. However, in order to apply Euler's formula we need to consider $\mathcal{A}(L)$ as a planar graph and take the symbolic "infinite" vertex into account. Therefore,

$$f = 2 - \left(\binom{n}{2} + 1\right) + n^2 = 1 + \frac{1}{2}(2n^2 - n(n-1)) = 1 + \frac{1}{2}(n^2 + n) = 1 + \binom{n}{2} + n\,.$$

□

The **complexity** of an arrangement is simply the total number of vertices, edges, and faces (in general, cells of any dimension).

Exercise 9.6 *Consider a set of lines in the plane with no three intersecting in a common point. Form a graph G whose vertices are the intersection points of the lines and such that two vertices are adjacent if and only if they appear consecutively along one of the lines. Prove that $\chi(G) \leqslant 3$, where $\chi(G)$ denotes the chromatic number of the graph G. In other words, show how to color the vertices of G using at most three colors such that no two adjacent vertices have the same color.*

9.2 Construction

As the complexity of a line arrangement is quadratic, there is no need to look for a subquadratic algorithm to construct it. We will simply construct it incrementally, inserting the lines one by one. Let $\ell_1, \dots, \ell_n$ be the order of insertion.

At Step i of the construction, locate ℓ_i in the leftmost cell of $\mathcal{A}(\{\ell_1, \dots, \ell_{i-1}\})$ it intersects. (The halfedges leaving the infinite vertex are ordered by slope.) This takes $O(i)$ time. Then traverse the boundary of the face F found until the halfedge h is found where ℓ_i leaves F (see Figure 9.2 for illustration). Insert a new vertex at this point, splitting F and h and continue in the same way with the face on the other side of h.

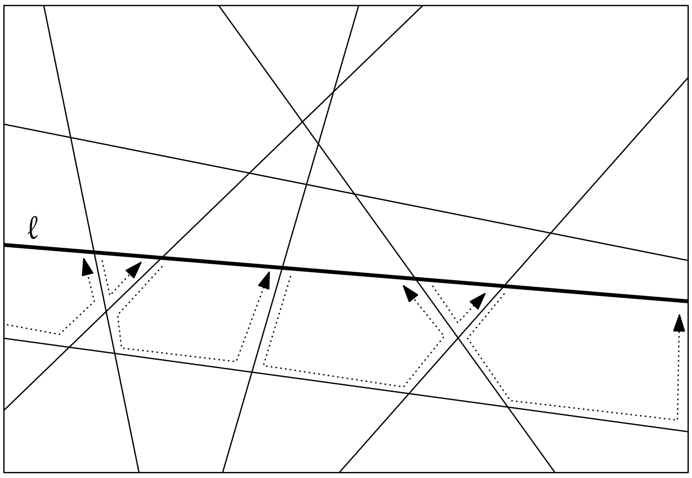


Figure 9.2: *Incremental construction: Insertion of a line ℓ. (Only part of the arrangement is shown in order to increase readability.)*

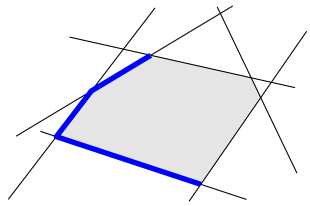
The insertion of a new vertex involves splitting two halfedges and thus is a constant time operation. But what is the time needed for the traversal? The complexity of $\mathcal{A}(\{\ell_1, \dots, \ell_{i-1}\})$ is $\Theta(i^2)$, but we will see that the region traversed by a single line has linear complexity only.

9.3 Zone Theorem

For a line ℓ and an arrangement $\mathcal{A}(L)$, the **zone** $Z_{\mathcal{A}(L)}(\ell)$ of ℓ in $\mathcal{A}(L)$ is the set of cells from $\mathcal{A}(L)$ whose closure intersects ℓ.

Theorem 9.7 *Given an arrangement $\mathcal{A}(L)$ of n lines in $\mathbb{R}^2$ and a line ℓ (not necessarily from L), the total number of edges in all cells of the zone $Z_{\mathcal{A}(L)}(\ell)$ is at most $6n$.*

Proof. Without loss of generality suppose that ℓ is horizontal (rotate the plane accordingly).



For each cell of $Z_{\mathcal{A}(L)}(\ell)$ split its boundary at its topmost vertex and at its bottommost vertex and orient all edges from bottom to top, horizontal edges from left to right. Those edges that have the cell to their right are called *left-bounding* for the cell and those edges that have the cell to their left are called right-bounding. For instance, for the cell depicted to the right all left-bounding edges are shown blue and bold.

We will show that there are at most $3n$ left-bounding edges in $Z_{\mathcal{A}(L)}(\ell)$ by induction on n. By symmetry, the same bound holds also for the number of right-bounding edges in $Z_{\mathcal{A}(L)}(\ell)$.

For $n = 1$, there is at most one (exactly one, unless ℓ is parallel to and lies above the only line in L) left-bounding edge in $Z_{\mathcal{A}(L)}(\ell)$ and $1 \leqslant 3n = 3$. Assume the statement is true for $n - 1$.

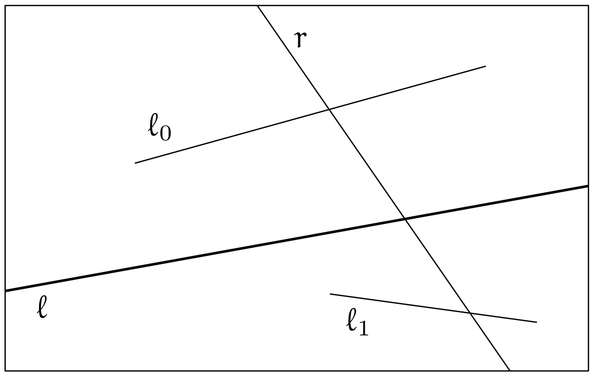


Figure 9.3: *At most three new left-bounding edges are created by adding* r *to* $\mathcal{A}(L\backslash\{r\})$.

If no line from L intersects ℓ, then all lines in $L \cup \{\ell\}$ are horizontal and there is at most $1 < 3n$ left-bounding edge in $Z_{\mathcal{A}(L)}(\ell)$. Else consider the rightmost line r from L intersecting ℓ and the arrangement $\mathcal{A}(L \setminus \{r\})$. By the induction hypothesis there are at most $3n - 3$ left-bounding edges in $Z_{\mathcal{A}(L\setminus\{r\})}(\ell)$. Adding r back adds at most three new left-bounding edges: At most two edges (call them ℓ_0 and ℓ_1) of the rightmost cell of $Z_{\mathcal{A}(L\setminus\{r\})}(\ell)$ are intersected by r and thereby split in two. Both of these two edges may be left-bounding and thereby increase the number of left-bounding edges by at most two. In any case, r itself contributes exactly one more left-bounding edge to that cell. The line r cannot contribute a left-bounding edge to any cell other than the rightmost: to the left of r, the edges induced by r form right-bounding edges only and to the right of r all other cells touched by r (if any) are shielded away from ℓ by one of ℓ_0 or ℓ_1. Therefore, the total number of left-bounding edges in $Z_{\mathcal{A}(L)}(\ell)$ is bounded from above by $3 + 3n - 3 = 3n$. □

Corollary 9.8 *The arrangement of* n *lines in* $\mathbb{R}^2$ *can be constructed in optimal* $O(n^2)$ *time and space.*