other, and to get from source to target the robot has to pass through a sequence of three collinear holes in the door (suppose the doorway is sufficiently small compared to the length of the robot).

Exercise 9.15 *The* 3*-Sum' problem is defined as follows: given three sets* S_1, S_2, S_3 *of* n *integers each, are there* $a_1 \in S_1$*,* $a_2 \in S_2$*,* $a_3 \in S_3$ *such that* $a_1 + a_2 + a_3 = 0$*? Prove that the* 3*-Sum' problem and the* 3*-Sum problem as defined in the lecture (*$S_1 = S_2 = S_3$*) are equivalent, more precisely, that they are reducible to each other in subquadratic time.*

9.7 Ham Sandwich Theorem

Suppose two thieves have stolen a necklace that contains rubies and diamonds. Now it is time to distribute the prey. Both, of course, should get the same number of rubies and the same number of diamonds. On the other hand, it would be a pity to completely disintegrate the beautiful necklace. Hence they want to use as few cuts as possible to achieve a fair gem distribution.

To phrase the problem in a geometric (and somewhat more general) setting: Given two finite sets R and D of points, construct a line that bisects both sets, that is, in either halfplane defined by the line there are about half of the points from R and about half of the points from D. To solve this problem, we will make use of the concept of levels in arrangements.

Definition 9.16 *Consider an arrangement* $\mathcal{A}(L)$ *induced by a set* L *of* n *non-vertical lines in the plane. We say that a point* p *is on the* **k-level** *in* $\mathcal{A}(L)$ *if there are at most* $k-1$ *lines below and at most* $n-k$ *lines above* p*. The* 1*-level and the* n*-level are also referred to as* **lower** *and* **upper envelope***, respectively.*

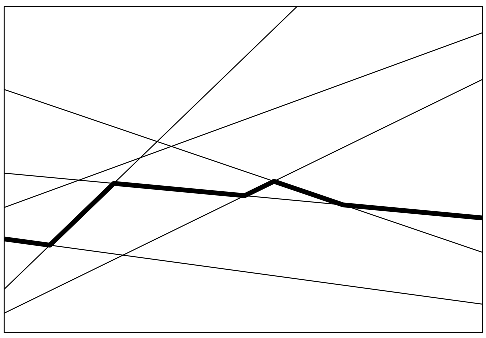

Figure 9.4: *The* 3*-level of an arrangement.*

Another way to look at the k-level is to consider the lines to be real functions; then the lower envelope is the pointwise minimum of those functions, and the k-level is defined by taking pointwise the k^{th}-smallest function value.

Theorem 9.17 *Let $R, D \subset \mathbb{R}^2$ be finite sets of points. Then there exists a line that bisects both R and D. That is, in either open halfplane defined by ℓ there are no more than $|R|/2$ points from R and no more than $|D|/2$ points from D.*

Proof. Without loss of generality suppose that both $|R|$ and $|D|$ are odd. (If, say, $|R|$ is even, simply remove an arbitrary point from R. Any bisector for the resulting set is also a bisector for R.) We may also suppose that no two points from $R \cup D$ have the same x-coordinate. (Otherwise, rotate the plane infinitesimally.)

Let R^* and D^* denote the set of lines dual to the points from R and D, respectively. Consider the arrangement $\mathcal{A}(R^*)$. The median level of $\mathcal{A}(R^*)$ defines the bisecting lines for R. As $|R| = |R^*|$ is odd, both the leftmost and the rightmost segment of this level are defined by the same line ℓ_r from R^*, the one with median slope. Similarly there is a corresponding line ℓ_d in $\mathcal{A}(D^*)$.

Since no two points from $R \cup D$ have the same x-coordinate, no two lines from $R^* \cup D^*$ have the same slope, and thus ℓ_r and ℓ_d intersect. Consequently, being piecewise linear continuous functions, the median level of $\mathcal{A}(R^*)$ and the median level of $\mathcal{A}(D^*)$ intersect (see Figure 9.5 for an example). Any point that lies on both median levels corresponds to a primal line that bisects both point sets simultaneously. □

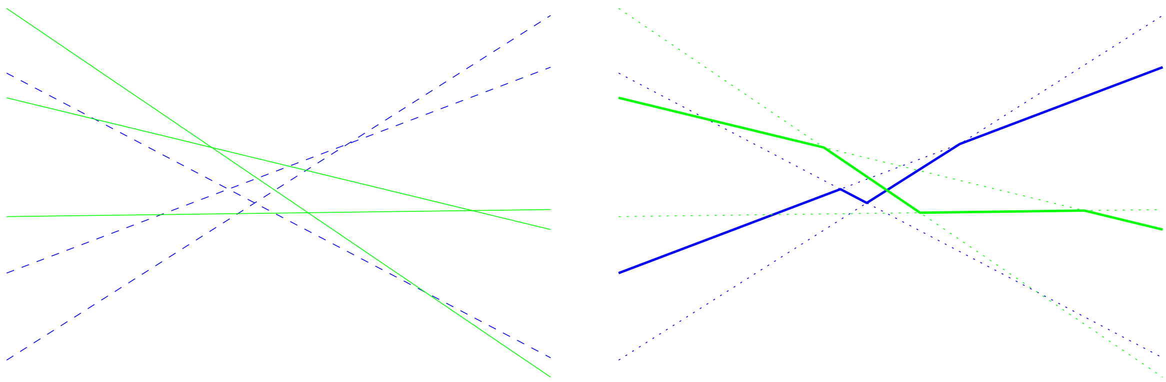

Figure 9.5: *An arrangement of 3 green lines (solid) and 3 blue lines (dashed) and their median levels (marked bold on the right hand side).*

How can the thieves use Theorem 9.17? If they are smart, they drape the necklace along some convex curve, say, a circle. Then by Theorem 9.17 there exists a line that simultaneously bisects the set of diamonds and the set of rubies. As any line intersects the circle at most twice, the necklace is cut at most twice. It is easy to turn the proof given above into an $O(n^2)$ algorithm to construct a line that simultaneously bisects both sets.

You can also think of the two point sets as a discrete distribution of a ham sandwich that is to be cut fairly, that is, in such a way that both parts have the same amount of ham and the same amount of bread. That is where the name "ham sandwich cut" comes from. The theorem generalizes both to higher dimension and to more general types of measures (here we study the discrete setting only where we simply count points). These generalizations can be proven using the *Borsuk-Ulam Theorem*, which states that any continuous map from S^d to $\mathbb{R}^d$ must map some pair of antipodal points to the same point. For a proof of both theorems see, for instance, Matoušek's book [8].

Theorem 9.18 *Let $P_1, \ldots, P_d \subset \mathbb{R}^d$ be finite sets of points. Then there exists a hyperplane H that simultaneously bisects all of $P_1, \ldots, P_d$. That is, in either open halfspace defined by H there are no more than $|P_i|/2$ points from P_i, for every $i \in \{1, \ldots, d\}$.*

This implies that the thieves can fairly distribute a necklace consisting of d types of gems using at most d cuts.

In the plane, a ham sandwich cut can be found in linear time using a sophisticated prune and search algorithm by Lo, Matoušek and Steiger [7]. But in higher dimension, the algorithmic problem gets harder. In fact, already for $\mathbb{R}^3$ the complexity of finding a ham sandwich cut is wide open: The best algorithm known, from the same paper by Lo et al. [7], has runtime $O(n^{3/2} \log^2 n / \log^* n)$ and no non-trivial lower bound is known. If the dimension d is not fixed, it is both NP-hard and W[1]-hard[4] in d to decide the following question [6]: Given $d \in \mathbb{N}$, finite point sets $P_1, \ldots, P_d \subset \mathbb{R}^d$, and a point $p \in \bigcup_{i=1}^{d} P_i$, is there a ham sandwich cut through p?

Exercise 9.19 *The goal of this exercise is to develop a data structure for halfspace range counting.*

a) Given a set $P \subset \mathbb{R}^2$ of n points in general position, show that it is possible to partition this set by two lines such that each region contains at most $\lceil \frac{n}{4} \rceil$ points.

b) Design a data structure of size $O(n)$, which can be constructed in time $O(n \log n)$ and allows you, for any halfspace h, to output the number of points $|P \cap h|$ of P contained in this halfspace h in time $O(n^\alpha)$, for some $0 < \alpha < 1$.

Exercise 9.20 *Prove or disprove the following statement: Given three finite sets A, B, C of points in the plane, there is always a circle or a line that bisects A, B and C simultaneously (that is, no more than half of the points of each set are inside or outside the circle or on either side of the line, respectively).*

Questions

39. *How can one construct an arrangement of lines in $\mathbb{R}^2$?* Describe the incremental algorithm and prove that its time complexity is quadratic in the number of lines (incl. statement and proof of the Zone Theorem).

[4] Essentially this means that it is unlikely to be solvable in time $O(f(d)p(n))$, for an arbitrary function f and a polynomial p.

40. *How can one test whether there are three collinear points in a set of* n *given points in* $\mathbb{R}^2$*?* Describe an $O(n^2)$ time algorithm.

41. *How can one compute the minimum area triangle spanned by three out of* n *given points in* $\mathbb{R}^2$*?* Describe an $O(n^2)$ time algorithm.

42. *What is a ham sandwich cut? Does it always exist? How to compute it?* State and prove the theorem about the existence of a ham sandwich cut in $\mathbb{R}^2$ and describe an $O(n^2)$ algorithm to compute it.

43. *Is there a subquadratic algorithm for General Position?* Explain the term 3-Sum hard and its implications and give the reduction from 3-Sum to General Position.

44. *Which problems are known to be 3-Sum-hard?* List at least three problems (other than 3-Sum) and briefly sketch the corresponding reductions.

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