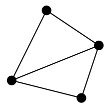


It is also possible to triangulate a geometric graph in linear time. But this problem is much more involved. Triangulating a single face of a geometric graph amounts to what is called "triangulating a simple polygon". This can be done in near-linear[3] time using standard techniques, and in linear time using Chazelle's famous algorithm, whose description spans a fourty pages paper [9].

Exercise 2.35. *We discussed the DCEL structure to represent plane graphs in Section 2.2.1. An alternative way to represent an embedding of a maximal planar graph is the following: For each triangle, store references to its three vertices and to its three neighboring triangles. Compare both approaches. Discuss different scenarios where you would prefer one over the other. In particular, analyze the space requirements of both.*

Connectivity serves as an important indicator for properties of planar graphs. Already Wagner showed that a 4-connected graph is planar if and only if it does not contain K_5 as a minor. That is, assuming 4-connectivity the second forbidden minor $K_{3,3}$ becomes "irrelevant". For subdivisions this is a different story. Independently Kelmans and Semour conjectured in the 1970s that 5-connectivity allows to consider K_5 subdivisions only. This conjecture was proven only recently[4] by Dawei He, Yan Wang, and Xingxing Yu.

Theorem 2.36 (He-Wang-Yu [18]). *Every 5-connected nonplanar graph contains a subdivision of* K_5.

Exercise 2.37. *Give a 4-connected nonplanar graph that does not contain a subdivision of* K_5.

Another example that illustrates the importance of the parameter connectivity is the following famous theorem of Tutte that provides a sufficient condition for Hamiltonicity.

Theorem 2.38 (Tutte [32]). *Every 4-connected planar graph is Hamiltonian.*

Moreover, for a given 4-connected planar graph a Hamiltonian cycle can also be computed in linear time [10].

2.5 Compact Straight-Line Drawings

As a next step we consider plane embeddings in the geometric setting, where every edge is drawn as a straight-line segment. A classical theorem of Wagner and Fáry states that this is not a restriction in terms of plane embeddability.

Theorem 2.39 (Fáry [13], Wagner [33]). *Every planar graph has a plane straight-line embedding.*

[3] $O(n \log n)$ or—using more elaborate tools—$O(n \log^* n)$ time

[4] The result was announced in 2015 and published in 2020.

This statement is quite surprising, considering how much more freedom arbitrarily complex Jordan arcs allow compared to line segments, which are completely determined by their endpoints. In order to further increase the level of appreciation, let us note that a similar "straightening" is not possible when fixing the point set on which the vertices are to be embedded: Pach and Wenger [27] showed that for a given planar graph G on n vertices $v_1, \ldots, v_n$ and a given set $\{p_1, \ldots, p_n\} \subset \mathbb{R}^2$ of n points, one can always find a plane embedding of G such that $v_i \mapsto p_i$, for all $i \in \{1, \ldots, n\}$. However, this is not possible in general with a plane *straight-line* embedding. For instance, K_4 does not admit a plane straight-line embedding on a set of points that form a convex quadrilateral, such as a rectangle. In fact, it is NP-hard to decide whether a given planar graph admits a plane straight-line embedding on a given point set [7].

Exercise 2.40. **a)** *Show that for every natural number $n \geqslant 4$ there exist a planar graph G on n vertices and a set $P \subset \mathbb{R}^2$ of n points in general position (no three points are collinear) so that G does not admit a plane straight-line embedding on P.*

b) *Show that for every natural number $n \geqslant 6$ there exist a planar graph G on n vertices and a set $P \subset \mathbb{R}^2$ of n points so that (1) P is in general position (no three points are collinear); (2) P has a triangular convex hull (that is, there are three points in P that form a triangle that contains all other points from P); and (3) G does not admit a plane straight-line embedding on P.*

Exercise 2.41. *Show that for every set $P \subset \mathbb{R}^2$ of $n \geqslant 3$ in general position (no three points are collinear) the cycle C_n on n vertices admits a plane straight-line embedding on P.*

Although Fáry-Wagner's theorem has a nice inductive proof, we will not discuss it here. Instead we will prove a stronger statement that implies Theorem 2.39.

A very nice property of straight-line embeddings is that they are easy to represent: We need to store points/coordinates for the vertices only. From an algorithmic and complexity point of view the space needed by such a representation is important because it appears in the input and output size of algorithms that work on embedded graphs. While the Fáry-Wagner Theorem guarantees the existence of a plane straight-line embedding for every planar graph, it does not provide bounds on the size of the coordinates used in the representation. But the following strengthening provides such bounds, by describing an algorithm that embeds (without crossings) a given planar graph on a linear size integer grid.

Theorem 2.42 (de Fraysseix, Pach, Pollack [15]). *Every planar graph on $n \geqslant 3$ vertices has a plane straight-line drawing on the $(2n-3) \times (n-1)$ integer grid.*

2.5.1 Canonical Orderings

The key concept behind the algorithm is the notion of a canonical ordering, which is a vertex order that allows to construct a plane drawing in a natural (hence canonical) way.

Reading it backwards one may think of a shelling or peeling order that destructs the graph vertex by vertex from the outside. A canonical ordering also provides a succinct representation for the combinatorial embedding.

Definition 2.43. *A plane graph is* **internally triangulated** *if it is biconnected and every bounded face is a (topological) triangle. Let* G *be an internally triangulated plane graph and* $C_\circ(G)$ *its outer cycle. A permutation* $\pi = (v_1, v_2, \ldots, v_n)$ *of* $V(G)$ *is a* **canonical ordering** *for* G *if it satisfies the following three conditions:*

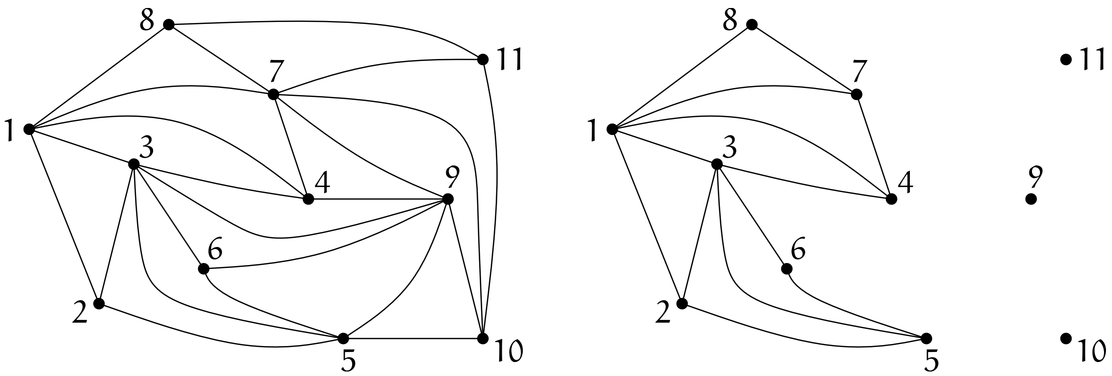
(CO1) G_k *is internally triangulated, for all* $k \in \{3, \ldots, n\}$*;*

(CO2) v_1v_2 *is on the outer cycle* $C_\circ(G_k)$ *of* G_k*, for all* $k \in \{3, \ldots, n\}$*; and*

(CO3) v_{k+1} *is located in the outer face of* G_k*, for all* $k \in \{3, \ldots, n-1\}$*;*

where G_k *is the subgraph of* G *induced by* $v_1, \ldots, v_k$*.*

Figure 2.18 shows an example. Note that there are permutations that do not correspond to a canonical order: for instance, when choosing the vertex 4 as the next vertex to be removed in Figure 2.18b, the resulting graph $G'_7 = G[\{1, 2, 3, 5, 6, 7, 8\}]$ is not biconnected (because 1 is a cut-vertex).



(a) G. **(b)** G_8.

Figure 2.18: *An internally triangulated plane graph with a canonical ordering.*

Theorem 2.44. *For every internally triangulated plane graph* G *and every edge* v_1v_2 *on its outer cycle, there exists a canonical ordering for* G *that starts with* v_1, v_2*. Moreover, such an ordering can be computed in linear time.*

Proof. Induction on n, the number of vertices. For a triangle, any order suffices and so the statement holds. Hence consider an internally triangulated plane graph $G = (V, E)$ on $n \geqslant 4$ vertices. We claim that it is enough to select a vertex $v_n \notin \{v_1, v_2\}$ on $C_\circ(G)$ that is not incident to a chord of $C_\circ(G)$ and then apply induction on $G \setminus \{v_n\}$.

We will show later that such a vertex v_n always exists. First let us prove the claim. We need to argue that if v_n is selected as described

(i) the plane graph $G_{n-1} := G \setminus \{v_n\}$ is internally triangulated,

(ii) the given edge $\{v_1, v_2\}$ is on the outer cycle $C_\circ(G_{n-1})$ of G_{n-1}, and

(iii) we can extend the inductively obtained canonical ordering for G_{n-1} with v_n to obtain a canonical ordering for G.

Property (ii) is an immediate consequence of $v_n \notin \{v_1, v_2\}$. Regarding (iii) note that (CO1)–(CO3) hold for $k = n$: The first two by assumption of the theorem (where we assume that G is internally triangulated and that v_1v_2 is an edge of its outer cycle), and (CO3) is trivial (because it applies to $k \leqslant n-1$ only). Hence to be able to apply induction it suffices to show (i).

The way G_{n-1} is obtained from G, every bounded face f of G_{n-1} also appears as a bounded face of G. As G is internally triangulated, f is a triangle. It remains to show that G_{n-1} is biconnected.

Consider the circular sequence of neighbors around v_n in G and break it into a linear sequence $u_1, \ldots, u_m$, for some $m \geqslant 2$, that starts and ends with the neighbors of v_n in $C_\circ(G)$. As G is internally triangulated, each of the bounded faces spanned by v_n, u_i, u_{i+1}, for $i \in \{1, \ldots, m-1\}$, is a triangle and hence $\{u_i, u_{i+1}\} \in E$. The outer cycle $C_\circ(G_{n-1})$ of G_{n-1} is obtained from $C_\circ(G)$ by removing v_n and replacing it with the (possibly empty) sequence $u_2, \ldots, u_{m-1}$. As v_n is not incident to a chord of $C_\circ(G)$ (and so neither of $u_2, \ldots, u_{m-1}$ appeared along $C_\circ(G)$ already), the resulting sequence forms a cycle, indeed. Add a new vertex v in the outer face of G_{n-1} and connect v to every vertex of $C_\circ(G_{n-1})$ to obtain a maximal planar graph $H \supset G_{n-1}$. By Theorem 2.32 the graph H is 3-connected and so G_{n-1} is biconnected, as desired. This also completes the proof of the claim.

Next let us show that we can always find a vertex $v_n \notin \{v_1, v_2\}$ on $C_\circ(G)$ that is not incident to a chord of $C_\circ(G)$. If $C_\circ(G)$ does not have any chord, this is obvious because every cycle has at least three vertices, one of which is neither v_1 nor v_2. So suppose that $C_\circ(G)$ has a chord c. The endpoints of c split $C_\circ(G)$ into two paths, one of which does not have v_1 nor v_2 as an internal vertex. We call this path the path *associated* to c. (Such a path has at least two edges because there is always at least one vertex "behind" a chord.) Among all chords of $C_\circ(G)$ we select c such that its associated path has minimal length. Then by this choice of c its associated path together with c forms an induced cycle in G. In particular, none of the (at least one) interior vertices of the path associated to c is incident to a chord of $C_\circ(G)$ because such a chord would either cross c or it would have an associated path that is strictly shorter than the one associated to c. So we can select v_n from these vertices. By definition the path associated to c does not contain v_1 nor v_2, hence this procedure does not select either of these vertices.

Regarding the runtime bound, we maintain for each vertex v whether it is on the current outer cycle and what is the number of incident chords with respect to the current outer cycle. Given a combinatorial embedding of G, it is straighforward to initialize this information in linear time. (Every edge is considered at most twice, once for each endpoint on the outer cycle.) We also maintain an unordered list of the *eligible* vertices, that is, those vertices that are on the outer cycle and not incident to any chord. This list is straightforward to maintain: Whenever a vertex information is updated, check before and after the update whether it is eligible and correspondingly add it to or remove it from the list of eligible vertices. We store with each vertex a pointer to its position in the list (*nil* if it is not eligible currently) so that we can remove it from the list in constant time if needed.

When removing a vertex v_n from G, there are two cases: Either v_n has two neighbors u_1 and u_2 only (Figure 2.19a), in which case the edge u_1u_2 ceases to be a chord. Thus, the chord count for u_1 and u_2 has to be decremented by one. Otherwise, there are $m \geqslant 3$ neighbors $u_1, \ldots, n_m$ (Figure 2.19b) and (1) all vertices $u_2, \ldots, u_{m-1}$ are new on the outer cycle, and (2) every edge incident to u_i, for $i \in \{2, \ldots, m-1\}$, and some other vertex on the outer cycle other than u_{i-1} or u_{i+1} is a new chord. These latter changes have to be reflected in the chord counters at the vertices. So to update these counters, we inspect all edges incident to one of $u_2, \ldots, u_{m-1}$. For each such edge, we check whether the other endpoint is on the outer cycle and, if so, increment the counter.

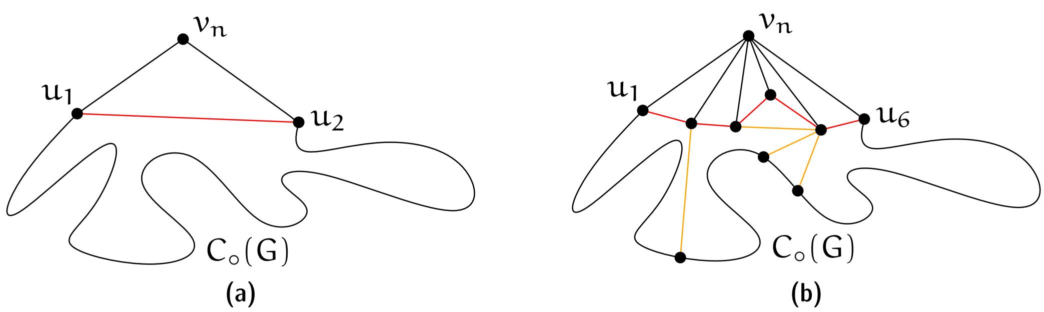


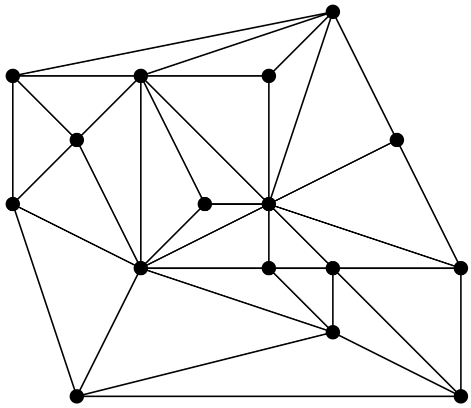
Figure 2.19: *Processing a vertex when computing a canonical ordering.*

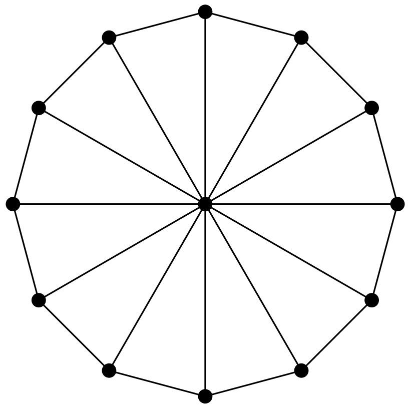
During the course of the algorithm every vertex appears once as a new vertex on the outer cycle. At this point all incident edges (in the current graph G_i) are examined. Similarly, when a vertex v_k is removed from G_K, all edges incident to v_k in G_k are inspected; and each vertex is removed at most once. Therefore, every edge is inspected at most three times: when one of its two endpoints appears first on the outer cycle, and when the first endpoint (and therefore the edge) is removed. Altogether this takes linear time because the number of edges in G is linear by Corollary 2.5. □

Using one of the linear time planarity testing algorithms, we can obtain a combinatorial embedding for a given maximal planar graph G. As every maximal plane graph is internally triangulated, we can then use Theorem 2.44 to provide us with a canonical ordering for G, in overall linear time.

Corollary 2.45. *Every maximal planar graph admits a canonical ordering. Moreover, such an ordering can be computed in linear time.* □

Exercise 2.46. *(a) Compute a canonical ordering for the following internally triangulated plane graphs:*





(b) *Give an infinite family of internally triangulated plane graphs* G_n *on* $n = 2k$ *vertices with at least* $k!$ *canonical orderings.*

(c) *Give an infinite family of internally triangulated plane graphs that have a unique canonical ordering for a specific choice of the starting edge* v_1v_2*.*

Exercise 2.47. (a) *Describe a plane graph* G *with* n *vertices that can be embedded (while preserving the outer face) on a grid of size* $(2n/3 - 1) \times (2n/3 - 1)$ *but not on a smaller grid.*

(b) *Can you draw* G *on a smaller grid if you are allowed to change the embedding?*

As simple as they may appear, canonical orderings are a powerful and versatile tool to work with plane graphs. As an example, consider the following partitioning theorem.

Theorem 2.48 (Schnyder [30]). *For every maximal planar graph* G *on at least three vertices and every fixed face* f *of* G*, the multigraph obtained from* G *by doubling the (three) edges of* f *can be partitioned into three spanning trees.*

Exercise 2.49. *Prove Theorem 2.48.* Hint: *Take a canonical ordering and build one tree by taking for every vertex* v_k *the edge to its first neighbor on the outer cycle* $C_\circ(G_{k-1})$*.*

Of a similar flavor is the following question.

Problem 2.50 (In memoriam Ferran Hurtado (1951–2014)).
Can every complete geometric graph on $n = 2k$ vertices (the complete straight line graph on a set of n points in general position) be partitioned into k plane spanning trees?

There are several positive results for special point sets [1, 5], and it is also known that there are always $\lfloor n/3 \rfloor$ edge disjoint plane spanning trees [4]. The general statement above has been refuted very recently [26]. However, it remains open if there always exists a partition into $k + 1$ plane spanning trees—or more generally, what is the minimum number of plane spanning trees that always suffices.