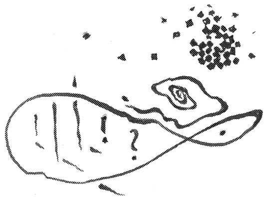

A little progress would be done in the world
if we were always afraid of possible negative consequences.

Georg Christoph Lichtenberg

Chapter 3

Infinity Is Not Equal Infinity or Why Infinity Is Infinitely Important in Computer Science

3.1 Why Do We Need Infinity?

The known universe is finite and the most physical theories consider the world to be finite. Everything we see and each object we touch is finite. Whatever we do in reality, we get in contact with finite things only.

Why to deal with infinity?
Is infinity not something artificial, simply a toy of mathematics?

In spite of possible doubts appearing in the first meetings with the concept of infinity, we allow us to claim that infinity is an unavoidable instrument for a successful investigation of our finite world. We already touch infinity for the first time in the basic school, where we meet the set

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

of natural numbers (nonnegative integers). The concept of this set can be formulated as follows:

> For each natural number i, there is a larger natural number $i + 1$.

In other words, there does not exist any number that would be larger than all other numbers (i.e., there exists no largest number), because for each number x there are numbers larger than x. What is the consequence of this concept? We are unable to write down the list of all natural numbers. It does not matter how many we have already written, there are still many missing. Hence, our writing is a never-ending story and because of this we speak about **potential infinity** or about **unbounded** number of natural numbers. We have a similar situation with the concept (the notion) of a line in geometry. Any line is potentially infinite and so its length is unbounded (infinitely large). One can walk along a line for an arbitrary long time. One never reaches the end and it does not matter which point (position) of the line you have reached, you can always continue in the walk in the same direction you used to reach this point.

The main trouble with understanding of the concept of infinity is that we are not capable to imagine any infinite object at once. We simply cannot see **actual infinity**. We realize that we have infinitely (unboundedly) many natural numbers, but we are not able to see all natural numbers at once. Similarly as we are unable to see a whole infinite line at once. We are always only able to view a finite fraction (part) of an infinite object. The way out we use is to denote infinite objects by symbols and then to work with these symbols as finite representations of the corresponding infinite objects.

To omit infinity, one can propose to exchange unbounded sizes by a huge finite bound. For instance, one can take the number[1] of all protons in the Universe as the largest number and to forbid all

[1] This number consists of 79 decimal digits

larger numbers. For most calculations and considerations one can be successful with this strategy. But not, if you try to compute the whole energy of the Universe or if you want to investigate all possible relations between particles of the Universe. It does not matter what huge number one chooses as the largest allowed number, there appear reasonable situations, whose investigation requires to perform calculations with numbers larger than the upper bound posed. Moreover, for every number x, we are not only aware of the existence of a larger number than x, we are even able to write this larger number and see it as a concrete object. Why should we forbid something what we can imagine (and so what has a concrete representation in our mind) and what we may even need?

To convince the reader in the usefulness of the concept of infinity, we need to provide more arguments than presenting the natural existence of potential infinity. We claim that by means of the concept of infinity we are able to investigate the world more successfully than without and so that infinity contributes to a better understanding of the finite world around. Infinity does not only enable to deal with infinitely large sizes. One can also consider infinitely small sizes.

> *What is the smallest positive rational number, i.e., what is the smallest positive fraction larger than 0?*

Consider the fraction $1/1000$. We can halve it and we get the fraction $1/2000$ that is smaller than $1/1000$. Now we can halve the resulting fraction again and get $1/4000 \ldots$. It does matter which small positive fraction

$$\frac{1}{x}$$

one takes, by halving it one gets the positive fraction

$$\frac{1}{2x}.$$

This fraction $1/2x$ is smaller than $1/x$ and surely still larger than 0. We see that this procedure of creating smaller and smaller numbers does not have any end too. For each positive number, there exists a smaller positive number, etc.

David Hilbert (1862 – 1943), one of the most famous mathematicians, said:

> *"In some sense, the mathematical analysis is nothing else than a symphony about the topic of infinity."*

And we add to this quotation that current physics as we know it would not exist without the notion of infinity. The key concepts and notions of mathematics such as derivation, limit, integral and differential equations would not exist without infinity. How can physics model the world without these notions? Unimaginable. One would already get troubles by building fundamental notions of physics. How can one define acceleration without these mathematical concepts? Many of the notions and concepts of mathematics were created because physics had a strong need to introduce and to use them.

The conclusion is that large parts of mathematics would disappear if infinity is forbidden to use. Since mathematics is the formal language of science and we often measure a degree of "maturity" of scientific disciplines with respect of using this language, the exclusion of the notion of infinity would set science several hundred years back.

We have the same situation in computer science. We have to distinguish between programs (that allow infinite computations) and algorithms that guarantee a finite computation on each input. There are infinitely many programs and infinitely many algorithmic tasks. A typical computing problem consists of infinitely many problem instances. Infinity is everywhere in computer science and so computer scientists cannot live without.

The goal of this chapter is not only to show that the concept of infinity is a research instrument of computer science. It will be strange because we do not satisfy ourselves with troubles appearing when dealing with potential infinity and actual infinity (that nobody has ever seen). We will still continue to pose the following strange question:

> *"Does there exist only the infinity or do there exist several differently large infinities?"*

Dealing with this, at the first glance stupid and too abstract question, was and is of enormous usefulness for science. Here we follow some of the most important discoveries about infinity in order to show that there exist at least two differently sized infinities. What is the gain of this? We can use this knowledge to show that the number of algorithmic problems (computing tasks) is larger than the number of all programs. In this way we obtain the first fundamental discovery of computer science.

> *One cannot automate everything. There are tasks, for which no algorithm exists and so which cannot be automatically solved by any computer or robot.*

Due to this discovery we are able to present in the next chapter concrete problems from practice that are not algorithmically (automatically) solvable. This is a wonderful example showing how the concept of an object that does not exist in the real world can help to achieve results and discoveries that are of practical importance. Remember! The way of using hypothetical and abstract objects in research is more typical than exceptional. And the most important thing that counts is whether the research goal was achieved.

3.2 The Concept of Cantor for Comparing the Sizes of Infinite Sets

Comparing finite numbers is simple. All numbers lay at the real axes in the growing order from the left to the right. The smaller one of two numbers is always laying to the left of the other one (Fig. 3.1).

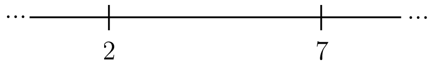

Fig. 3.1

Hence, 2 is smaller than 7 because the position of it is to the left to the position of 7 on the axes. But this is not a concept for comparing numbers because the numbers are a priori laid on the axes in such a way that they increase from the left to the right and decrease from the right to the left. Though the axes is infinite in both directions only finite numbers lay on it. It does not matter which position (which point) we consider, the number sitting there is always a concrete finite number. This is the concept of potential infinity. One can walk along the axes arbitrarily long to the right or to the left and each position reached on this trip contains a concrete finite number. There are no infinite numbers on the axes. To denote infinity in mathematics we use the symbol

$$\infty$$

called a "laying eight". Originally this symbol came from the letter Aleph of the Hebraic alphabet. But if one represents infinity by just one symbol ∞, there does not exist any possibility of comparing different infinities.

What do we need to overcome this trouble?

We need a new representation of numbers. To get it, we need the notion of a set. A set is any collection of objects (elements) that are pairwise distinct. For instance, $\{2, 3, 7\}$ is a set that contains three numbers $2, 3$, and 7. The set {John, Anna, Peter, Paula} contains four objects (elements) John, Anna, Peter, and Paula. For any set A, we use the notation

$$|A|$$

for the numbers of elements in A and call $|A|$ the **cardinality (size) of *A***. For instance,

$$|\{2, 3, 7\}| = 3 \text{ and } |\{\text{John, Anna, Peter, Paula}\}| = 4\ .$$

Now, we take the sizes of sets as representations of numbers. In this way the cardinality of the set $\{2, 3, 7\}$ represents the integer 3, and the cardinality of the set {John, Anna, Peter, Paula} represents the number 4. Clearly, every positive integer gets a lot of different representations in this way. For instance, all

$$|\{1, 2\}|\ ,\ |\{7, 11\}|\ ,\ |\{\text{Petra, Paula}\}|\ ,\ |\{\Box, \bigcirc\}|$$

are representations of the integer 2. Is it not fussy? What is the gain of this too complicated representation of integers?

May be you find this representation to be awkward for the comparison of finite numbers[2]. But using this way of representing numbers we obtain the possibility to compare infinite sizes. The cardinality

$$|\mathbb{N}|$$

for $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the infinite number that corresponds to the number of all natural numbers. If $\mathbb{Q}^+$ denotes the set of all positive rational numbers, then the number

$$|\mathbb{Q}^+|$$

represents the infinite number that corresponds to the number of all positive rational numbers (fractions). And

$$|\mathbb{R}|$$

is the infinite number that corresponds to the number of all real numbers, assuming $\mathbb{R}$ denotes the set of all real numbers. Now we see the gain. We are allowed to ask

"Is $|\mathbb{N}|$ smaller than $|\mathbb{R}|$?"

or

"Is $|\mathbb{Q}^+|$ smaller than $|\mathbb{R}|$?"

[2]With high probability, this is the original representation of natural numbers used by Stone Age man. Small children use first the representation of numbers by sets in order to develop later an abstract concept of a "number".

Due to this way of representing numbers we are able for the first time to pose the question whether an infinity is larger than another infinity.

We have reduced our problem of comparing (infinite) numbers to comparing sizes of (infinite) sets. But now the following question arises:

"How to compare the sizes of two sets?"

If the sets are finite, then the comparison is simple. One simply counts the number of elements in both sets and compares the corresponding cardinalities. For sure, we cannot do it for infinite sets in this way. If one tries to count the elements of infinite sets, then it would be a never-ending story and so the proper comparison would never be performed. Hence, we need a general method for comparing sizes of sets that would work for finite as well as infinite sets and that one could judge as reasonable and trustworthy. This means that we are again on the deepest axiomatic level of science. Our fundamental task is to create the notion of infinity and the definition of **"smaller or equal to"** for the comparison of the cardinalities of two sets.

Now we let a shepherd help us. This is no shame because mathematicians did it, too.

A shepherd has a big flock of sheep with many black and white sheep. He never used to go to school and though his wisdom (that does not allow him to leave the mountains) he can count only to five. He wants to find out whether he has more black sheep than white ones or vice versa (Fig. 3.2).

How can he do it without counting? In the following simple and genius way. He takes one black sheep and one white sheep and creates one pair

$$\text{(white sheep, black sheep)}\ ,$$

and sends them away from the flock. Then he creates another white-black pair and sends it away (Fig. 3.3). He continues in this way until he has sheep of one color only or there are no remaining

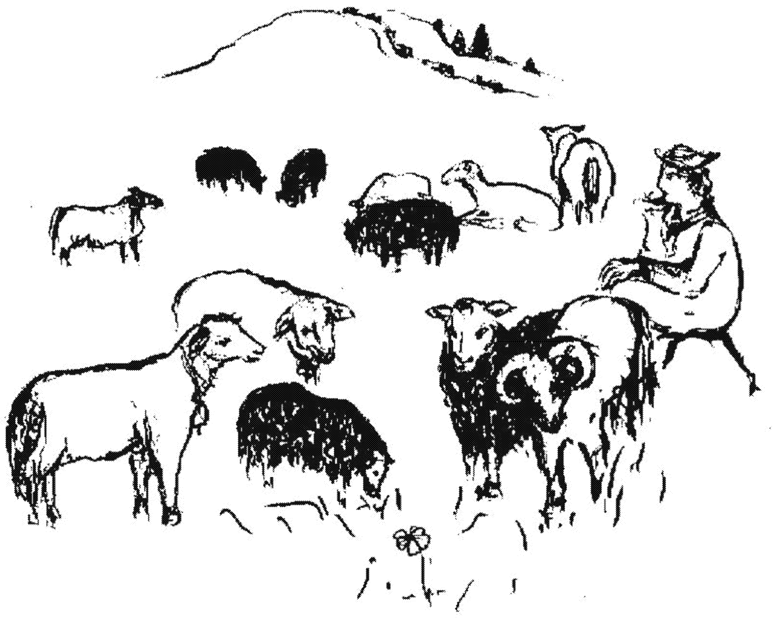

Fig. 3.2

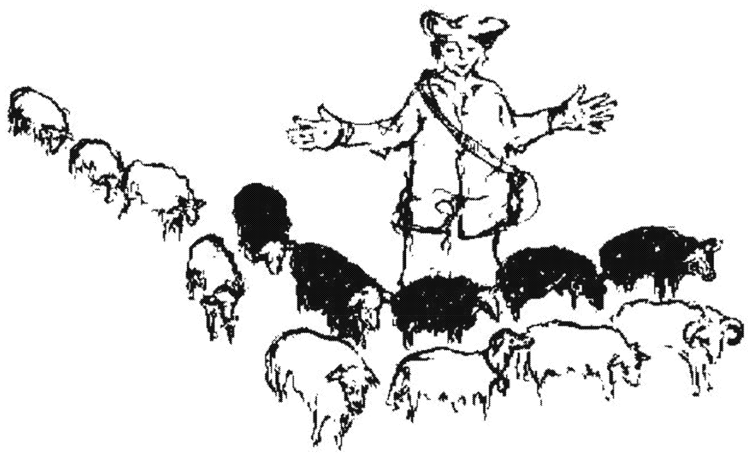

Fig. 3.3

sheep at all (i.e., until there is no possibility to build a white-black pair of sheep). Now he can make a conclusion in the following way:

(i) If no sheep remained, he has as many white sheep as black ones.

(ii) If one or more white sheep remained in the flock, then he has more white sheep than black ones (Fig. 3.3).

(iii) If one or more black sheep remained in the flock, then he has more black sheep than white ones.

Pairing of sheep and the conclusion (i) used mathematicians as the base for comparing the sizes of sets.

Definition 3.1 Let A and B be two sets. A **matching** of A and B is building a set of pairs (a, b) that satisfies the following rules:

(i) The element a belongs to A ($a \in A$) and the element b belongs to B ($b \in B$).

(ii) Each element of A is the first element of exactly one pair (i.e., no element of A is involved in two or more pairs and no element of A remains unmatched).

(iii) Each element of B is the second element of exactly one pair.

For each pair (a, b), we say that ***a* and *b* are married**. We say that ***A* and *B* have the same size** or that **the size of *A* equals to the size of *B*** and write

$$|\boldsymbol{A}| = |\boldsymbol{B}|$$

if there exists a matching of A and B. We say that **the size of *A* is not equal to the size of *B*** and write

$$|\boldsymbol{A}| \neq |\boldsymbol{B}|$$

if there does not exist any matching of A and B.

Consider the two sets $A = \{2, 3, 4, 5\}$ and $B = \{2, 5, 7, 11\}$ in Fig. 3.4. Fig. 3.4 depicts the matching

$$(2, 2), (3, 5), (4, 7), (5, 11) \ .$$

Each element of A is involved in exactly one pair of the matching as the first element. For instance, the element 4 from A is involved

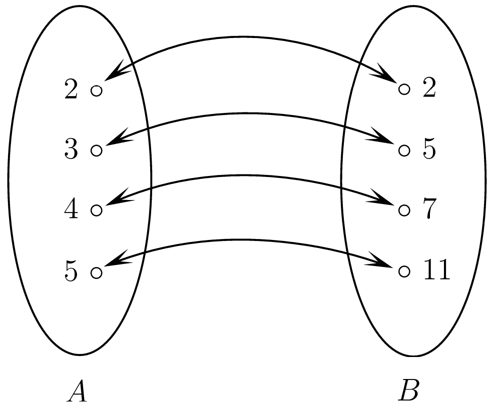

Fig. 3.4

as the first element of the third pair $(4, 7)$. Each element of B is involved in exactly one pair as the second element. For instance, the element 5 of B is involved in the second pair. In other words, each element of A is married to exactly one element of B, each element of B is married to exactly one element of A and so, no element of A or B remained single. Therefore, we can conclude

$$|\{2, 3, 4, 5\}| = |\{2, 5, 7, 11\}| \ .$$

You can also find out other matchings of A and B. For instance,

$$(2, 11), (3, 7), (4, 5), (5, 2)$$

is also a matching of A and B.

Exercise 3.1 a) Give two other matchings of the sets $A = \{2, 3, 4, 5\}$ and $B = \{2, 5, 7, 11\}$.
b) Why is $(2, 2), (4, 5), (5, 11), (2, 7)$ not a matching of A and B?

Following this concept of comparing the sizes of two sets, a set A of girls and a set of boys are equally sized, if one can get married all the women and men from A and B in such a way that no single remains[3].

[3]Building of homosexual pairs is not allowed here.

A matching of the sets $C = \{1, 2, 3\}$ and $D = \{2, 4, 6, 8\}$ cannot exist because every attempt to match the elements of D and C finishes in the situation, where one element of D remains single. Therefore, $|D| \neq |C|$ holds. An unsuccessful attempt to match C and D is depicted in Fig. 3.5.

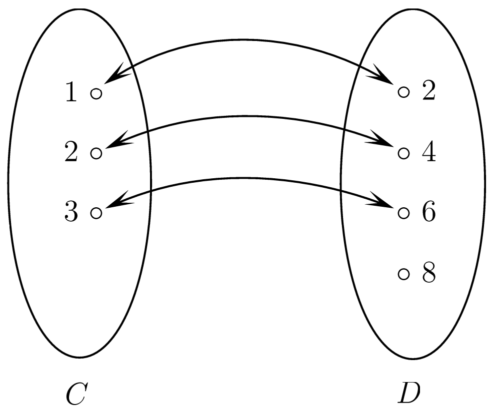

Fig. 3.5

Fig. 3.6 shows another attempt to match C and D. Here the result is not a matching of C and D because the element 3 of C is married to two elements 4 and 8 from D.

But we do not need the concept of matching in order to compare the sizes of finite sets. We were able to do it also without this concept. In the previous part, we only checked that our matching concept does work in the finite world[4]. In what follows we try to apply this concept for infinite sets. Consider the two sets

$$\mathbb{N}_{even} = \{0, 2, 4, 6, 8, \ldots\}$$

of all even natural numbers and

$$\mathbb{N}_{odd} = \{1, 3, 5, 7, 9, \ldots\}$$

[4]If the concept did not work in the finite world, then we have to reject it.

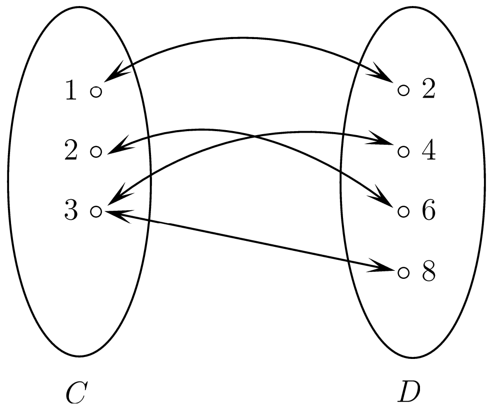

Fig. 3.6

of all odd natural numbers. At the first glance, these sets look to be of the same size and so we try to verify it by the means of our concept. We match each even number $2i$ the odd number $2i+1$.

Following Fig. 3.7 we see that we get an infinite sequence of pairs

$$(0,1),(2,3),(4,5),(6,7),\ldots,(2i,2i+1),\ldots$$

in this way. This sequence of pairs is a correct matching of A and B. No element from $\mathbb{N}_{even}$ or from $\mathbb{N}_{odd}$ is involved in two or more pairs (is married to more than one element). On the other hand no element remained single (unmarried). For each even number $2k$ from $\mathbb{N}_{even}$, we have the pair $(2k, 2k+1)$. For each odd number $2m+1$ from $\mathbb{N}_{odd}$, we have the pair $(2m, 2m+1)$. Hence, we verified that the equality $|\mathbb{N}_{even}| = |\mathbb{N}_{odd}|$ holds.

Exercise 3.2 Prove that $|\mathbb{Z}^+| = |\mathbb{Z}^-|$, where $\mathbb{Z}^+ = \{1,2,3,4,\ldots\}$ and $\mathbb{Z}^- = \{-1,-2,-3,-4,\ldots\}$. Draw a picture depicting your matching similarly as we did for $\mathbb{N}_{even}$ and $\mathbb{N}_{odd}$ in Fig. 3.7.

Up to this point everything looks tidy, understandable, and acceptable. Now, we present something which may be hardly come to term with, at least at the first attempt. Consider the sets

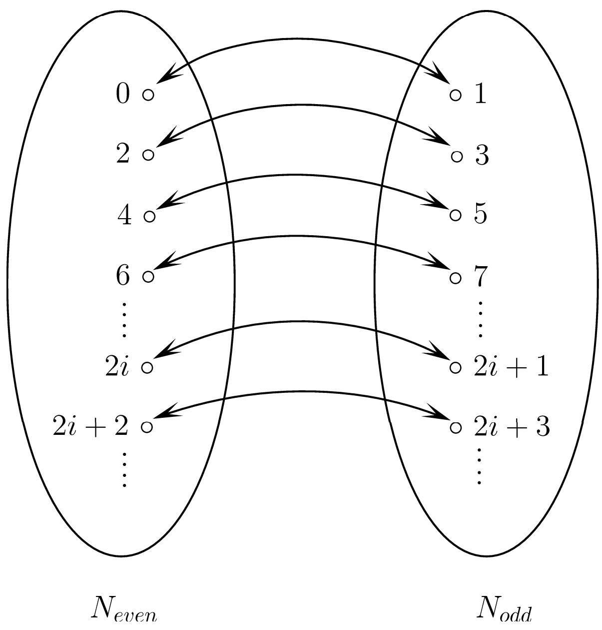

Fig. 3.7

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\} \text{ and } \mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\} \ .$$

All elements of $\mathbb{Z}^+$ are in $\mathbb{N}$, and so

$$\mathbb{Z}^+ \subseteq \mathbb{N} \ ,$$

i.e., $\mathbb{Z}^+$ is a **subset** of $\mathbb{N}$. Moreover, the element 0 belongs to $\mathbb{N}$ ($0 \in \mathbb{N}$), but not to $\mathbb{Z}^+$ ($0 \notin \mathbb{Z}^+$). We therefore say that $\mathbb{Z}^+$ is a **proper subset** of $\mathbb{N}$ and write $\mathbb{Z}^+ \subset \mathbb{N}$. The notion "$A$ is a proper subset of B" means that A is a part of B but not the whole B. We can see this situation transparently for the case

$$\mathbb{Z}^+ \subset \mathbb{N}$$

in Fig. 3.8. We see that $\mathbb{Z}$ is completely involved in $\mathbb{N}$ but $\mathbb{Z}$ does not cover the whole $\mathbb{N}$ because $0 \in \mathbb{N}$ and $0 \notin \mathbb{Z}^+$.

Though we claim that

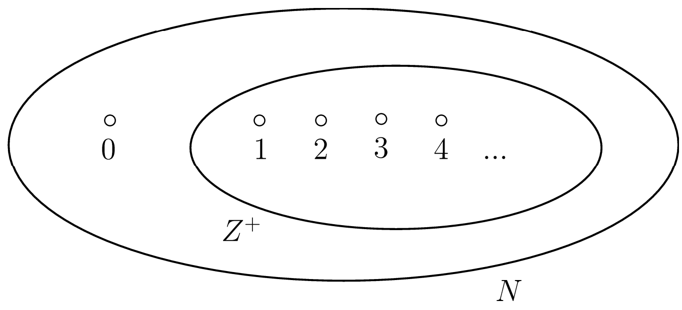

Fig. 3.8

$$|\mathbb{N}| = \left|\mathbb{Z}^+\right|$$

is true, i.e., that the sizes of $|\mathbb{N}|$ and $|\mathbb{Z}^+|$ are equal to each other. We justify this claim by building the following matching

$$(0, 1), (1, 2), (2, 3), \ldots, (i, i + 1), \ldots\ ,$$

depicted in Fig. 3.9.

We clearly see that all elements of $\mathbb{N}$ and $\mathbb{Z}^+$ are correctly married. No element remains single. The conclusion is that $\mathbb{N}$ is not larger than $\mathbb{Z}^+$ though $\mathbb{N}$ has one more element than $\mathbb{Z}^+$. But this fact may not be too surprising or even worrying. It says only that

$$\infty + 1 = \infty\ ,$$

and so that increasing infinity by 1 does not lead to a larger infinity. This does not look surprising. What is 1 in comparison with infinity? A nothing that can be neglected. This, at the first glance, surprising combination of the facts

$$\mathbb{Z}^+ \subset \mathbb{N} \text{ (Fig. 3.8) and } \left|\mathbb{Z}^+\right| = |\mathbb{N}| \text{ (Fig. 3.9)}$$

provides the fundamentals used for creating the mathematical definition of infinity. Mathematicians needed thousands of years to find out this definition and then one generation exchange in research to be able to accept it and fully imagine its meaning. It

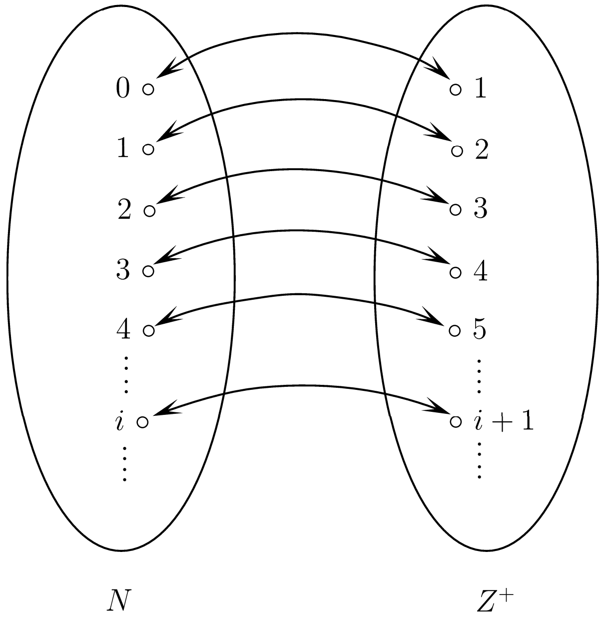

Fig. 3.9

was not so easy for them to see that this definition provides that, what they strived for, namely a formal criterion for distinguishing between finite sets and infinite sets.

Definition 3.2 A set A is infinite if and only if there exists a proper subset B of A such that

$$|A| = |B| \ .$$

In other words:

> *"An object is infinite if there is a proper part of the object that is as large as the whole object."*

Now you can say: *"Halt! This is too much for me. I cannot accept something like that. How can a part be of the same size as the whole? Something like this does not exist."*

I find it excellent. that you have this meaning. Especially because of this, this definition is good. In the real world in which everything is finite, no part can be as large as the whole. This is what we can agree on. No finite (real) object can have this strange property. And in this way, Definition 3.2 says correctly that all such objects are finite (i.e., not infinite). But in the artificial world of infinity, it is not only possible to have this property but even a duty. And so this property is exactly that, what we where searching for. Since who has this property is infinite and who does not have this property is finite. In this way, Definition 3.2 provides a criterion for classifying objects into finite ones and infinite ones and this is exactly what one expects from such a definition.

To get a deeper understanding of this at the first glance strange property of infinite objects, we present two examples.

Example 3.1 Hilbert hotel

Let us consider a hotel with infinitely many single rooms that is well known as Hilbert hotel. The rooms are enumerated as follows:

$$Z(0), Z(1), Z(2), Z(3), \ldots, Z(i), \ldots \quad .$$

All rooms are occupied, i.e. there is exactly one guest in each room. Now, a new guest is entering the hotel and asks the porter: "Do you have a free room for me?" "'No problem", answers the porter and accommodates the new guest by the following strategy. He asked every guest in the hotel to move in the next room with the number that is higher by 1 with respect to the number of the room used up till now. Following this request, the guest from room $Z(0)$ moves to the room $Z(1)$, the guest from $Z(1)$ moves to $Z(2)$, etc. In general, the guest from $Z(i)$ moves to the room $Z(i+1)$. In this way, the room $Z(0)$ becomes free and so $Z(0)$ can be assigned to the newcomer (Fig. 3.10).

We observe that, after the move, every guest has her or his own room and the room $Z(0)$ became free for the newcomer. Mathematicians argue for the truthfulness of this observation as follows. Clearly. the room $Z(0)$ is free after the move. The task is to show

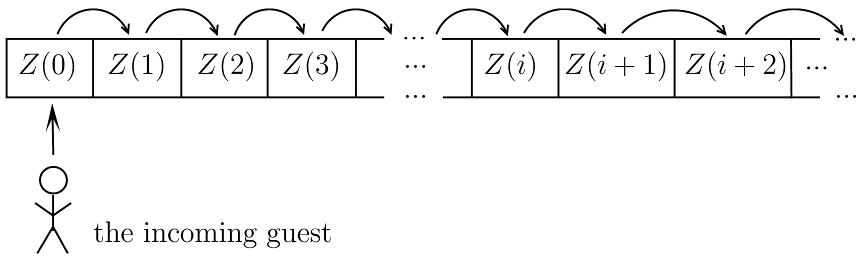

Fig. 3.10

that every guest has one's own room after the move. Let G be an arbitrary guest. This person G lives alone in a concrete room before the move. Let $Z(n)$ be the number of this room. Following the instruction of the porter the guest G moves from $Z(n)$ to $Z(n+1)$. He can do it because the room $Z(n+1)$ becomes free because the guest from this room moves to room $Z(n+2)$. Hence, after the moves guest G leaves alone in room $Z(n+1)$. Since our argumentation is valid for every guest of the hotel, all guests have a single room accommodation after the move.

The solution above shows why the actual infinity was considered as paradox[5] of mathematics for a long time. Hotel Hilbert is an actual infinity. Something like this can only be outlined by drawing a finite part of it and adding $\cdots$. But nobody can see it at once. Hence, it is not possible to observe the whole move of infinitely many guests at once. On the other hand, observing each particular guest separately, one can verify that the move successfully works.

Only when one was able to realize that infinity differs from finity by having proper subparts of the same size as the whole, this paradox was solved[6]. We observe that the move corresponds to matching the elements of the set $\mathbb{N}$ (as the set of guests) with the set $\mathbb{Z}^+$ (as the set of rooms up to the room $Z(1)$). □

[5] a contradictory fact or an inexplicable situation

[6] and so it is not a paradox anymore

Exercise 3.3 a) Three newcomers enter hotel Hilbert. As usual, the hotel is completely booked. Play the role of the porter and accommodate the three new guests in such a way that no former guest has to leave the hotel and after the move, each new guest and each former guest possesses one's own room. If possible, arrange all using one move of the guest instead of organizing 3 moves one after each other.

b) A newcomer enters hotel Hilbert and asks for his favored room $Z(7)$. How can the porter satisfy this request?

We take the next example from physics. Physicists discovered it as a remedy for depressions caused by imagining that our Earth and so mankind too is tiny in the comparison with the huge universe[7].

Example 3.2 Let us view our Earth and Universe as infinite sets of points of the size 0 that can lay arbitrarily close each to each other. To simplify our story we view everything two dimensionally instead of working in three dimensions. The whole Universe can be viewed as a large sheet of paper and Earth can be depicted as a small circle on the sheet (3.11). If somebody doubts about viewing our small Earth as an infinite set of points, remember that there are infinitely many points on the finite part of the real axes between the numbers 0 and 1. Each rational number between 0 and 1 can be viewed as a point on the line between 0 and 1. And there are infinitely many rational numbers between 0 and 1. We proved this fact already by generating infinitely many rational numbers between 0 and 1 in our unsuccessful attempt to find the smallest positive rational number.

Another justification of this fact is related to the proof of the following claim.

> For any two different rational numbers a and b, $a < b$, there are infinitely many rational numbers between a and b.

The first number between a and b we generate is the number $c_1 = \frac{a+b}{2}$, i.e., the average value of a and b. The next one is $c_2 = \frac{c_1+b}{2}$, i.e., the average of c_1 and b. In general, the i-th generated number from $[a, b]$ is

$$c_i = \frac{c_{i-1} + b}{2}$$

[7] In this way, physicists try to ease the negative consequences of their discoveries

, i.e., the average of c_{i-1} und b. When $a = 0$ and $b = 1$, then one gets the infinite sequence

$$\frac{1}{2}\,,\ \frac{3}{4}\,,\ \frac{7}{8}\,,\ \frac{15}{16}\,,\ \ldots$$

of pairwise different rational numbers between 0 and 1.

Now let us finally switch to the fact, physicists want to tell us. All points of our huge universe beyond of Earth can be matched with the points of Earth. This claim has two positive (healing) interpretations:

(i) The number of points of our Earth is equal to the number of points of Universe outside Earth.

(ii) Everything what happens in the Universe can be reflected on Earth and so imitated in our tiny world.

Hence, our task to search for a matching between the Earth's points and the points outside Earth. In what follows we show how to assign an Earth point P_E to any point P_U outside Earth.

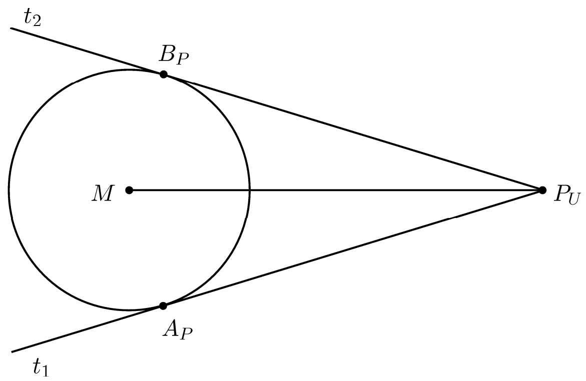

Fig. 3.11

First, we connect P_U and the Earth center M by a line (Fig. 3.11). The point P_E we are searching for has to lay on this line. Next, we depict the two tangents t_1 and t_2 of the circuit that goes through the point P_U (Fig. 3.11). Remember that a tangent of a circuit is a line that has exactly one common point with the circuit. We call by A_P the point in which t_1 touches the circuit and by B_P the common point[8] of the circuit and the line t_2 (see Fig. 3.11). Finally, we connect the points B_P and A_P by a line B_PA_P (Fig. 3.12). The point in the intersection of the lines B_PA_P and P_UM is the Earth point P_E we assign to P_U (Fig. 3.12).

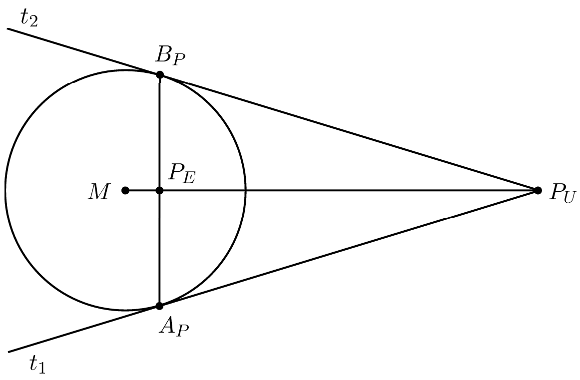

Fig. 3.12

Next, we have to show that this geometric assignment of P_E to P_U defines a matching between the Earth's points and the points outside Earth. Namely we have to show that one always assigns two distinct Earth's points P_E and P'_E to two different points P_U and P'_U outside Earth.

To verify this fact, we distinguish two possibilities with respect to the positions of P_U and P'_U according to M.

(i) The points M, P_U, and P'_U do not lay on the same line. This situation is depicted in Fig. 3.13. We know that P_E lies on the line MP_U and that P'_E lies on the line MP'_U. Since the only

[8] Mathematicians would say that the point A_P is the intersection of the circuit and t_1 and that B_P is the intersection of the circuit and t_2.

common point of the lines MP_U and MP'_U is M and M is different from P_E and P'_E, independently on the positions of P_E and P'_E on their lines, the points of P_E and P'_E must be different.

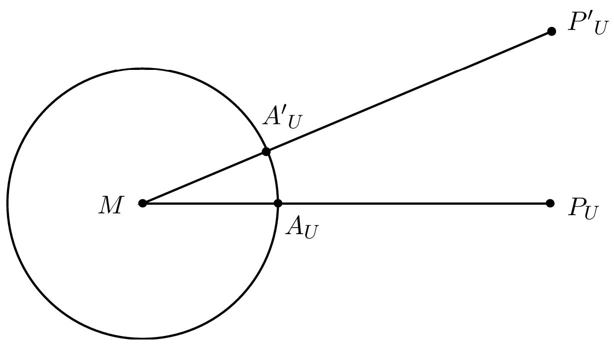

Fig. 3.13: E_U lies on MA_U and E'_U lies on MA'_U, and therefore E_U and E'_U are different points.

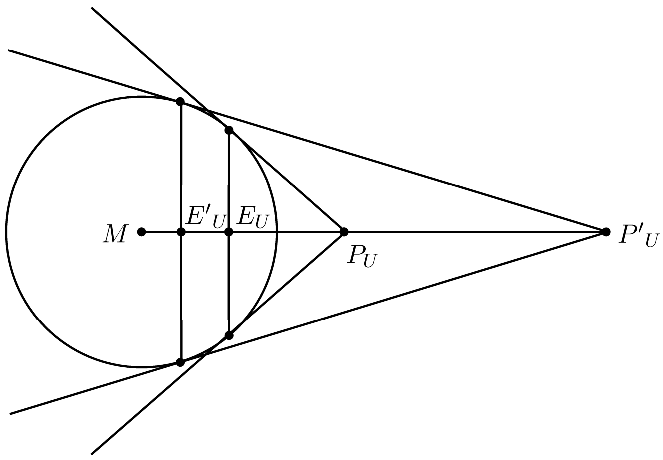

Fig. 3.14

(ii) All three points M, P_U, and P'_U lay on the same line (Fig. 3.14). Therefore, E_U and E'_U lay on this line, too. Then, we perform our assignment construction for both points P_U and P'_U as depicted in Fig. 3.12. We immediately see in Fig. 3.14 that E_U and E'_U are different.

We showed that, independently on the fact how many times Universe is larger than Earth, the number of points of Earth is equal to the number of points of Universe outside Earth. □

Exercise 3.4 Complete Fig. 3.13 by estimating the exact positions of points P_E and P'_E.

Exercise 3.5 Consider the semicircle in Fig. 3.15 and the line AB that is the diameter of the circuit. Justify geometrically as well as by calculations that the number of points of the line AB is the same as the number of points of the curve of the semicircle.

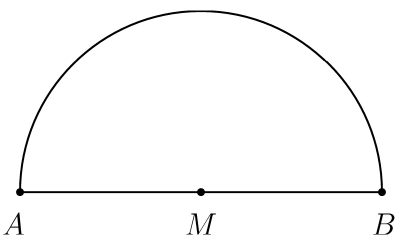

Fig. 3.15

Exercise 3.6 Consider the curve of the function F in Fig. 3.16 and the line AB. Why does this curve have as many points as the line AB?

If you still have stomachache when trying to imagine and to accept the concept of infinity, please, do not worry. Mathematicians needed many years to develop this concept and after discovering it 20 years were needed to get it accepted by broad mathematical community. Take time for repeated confrontations with the definition of infinite sets. Only if one iteratively deals with this topic, one can understand why one takes over this definition of the infinity as an axiom of mathematics and why mathematicians consider

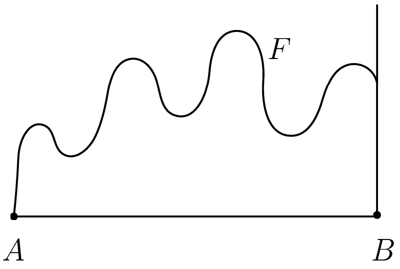

Fig. 3.16

it not only trustworthy, but they even do not see any alternative to this definition.

In what follows we shortly discuss the most frequent proposal for the concept of comparing infinite sizes that some listener proposed after the first confrontation with infinity. If

$$A \subset B$$

holds (i.e., if A is a proper subset of B), then

$$|A| < |B| \ .$$

Clearly, this attempt to compare infinite sizes reflects in another way the refusal of our key idea that a part of an infinite object may be as large as the whole. This proposal for an alternative definition has two drawbacks. First, one can use it only for comparing two such sets that one is a subset of the other. This definition does not provide the possibility to compare two different sets such as $\mathbb{Z}^- = \{-1, -2, -3, \ldots\}$ and $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. For a comparison of these two sets one has to search for another relation between them. Realizing this drawback, some listeners propose to accept the matching approach in the following way. One can find a matching between one of the sets and a subset of another one and then compare using the originally proposed subset principle. We show that one can get a nonsense in this way. Namely that

$$|\mathbb{N}| < |\mathbb{N}| \ ,$$

i.e., that $\mathbb{N}$ is smaller than $\mathbb{N}$ itself. Using the concept of matching we proved

$$|\mathbb{N}| = |\mathbb{Z}^+|. \tag{3.1}$$

Since $\mathbb{Z}^+ \subset \mathbb{N}$, using the subset principle, one gets

$$|\mathbb{Z}^+| < |\mathbb{N}|. \tag{3.2}$$

Combining (3.1) and (3.2) we obtain

$$|\mathbb{N}| = |\mathbb{Z}^+| < |\mathbb{N}|$$

and so $|\mathbb{N}| < |\mathbb{N}|$.

In this way we proved that the concept of the shepherd (of matching) and the subset principle for comparing the cardinalities of two sets contradict each other because adopting both at once leads to an obvious nonsense.

Why do we spend so much time to discuss this axiom of mathematics and why do we take so big effort to get its understanding? As you may already suspect, this axiom is only the beginning of our troubles. The concept of infinity is not the only surprise of this chapter. In some sense we showed $\infty = \infty + 1$ and also give to understand that $\infty = \infty + c$ for any finite number c. Example 3.2 and the following exercises even intimate

$$\infty = c \cdot \infty$$

for an arbitrary finite number (constant) c.

Let us consider $\mathbb{N}$ and the set

$$\mathbb{N}_{even} = \{0, 2, 4, 6, \ldots\} = \{2i \mid i \in \mathbb{N}\}$$

of all even natural numbers. At the first glance $\mathbb{N}$ contains twice as many elements as $\mathbb{N}_{even}$. In spite of this view (Fig. 3.17) one can match the elements of $\mathbb{N}$ and of $\mathbb{N}_{even}$ as follows:

$$(0,0),(1,2),(2,4),(3,6),\ldots,(i,2i),\ldots \ .$$

We see that each element of both sets is married exactly once. The immediate consequence is

$$|\mathbb{N}| = |\mathbb{N}_{even}| \ .$$

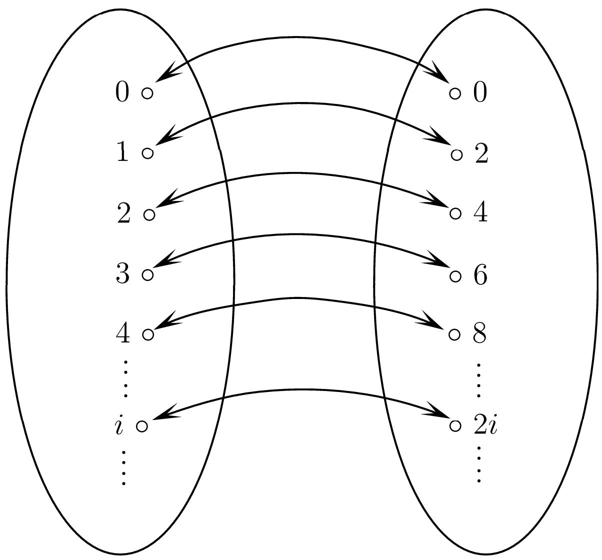

Fig. 3.17

We can show up this a little bit surprising result

$$2 \cdot \infty = \infty$$

again by a story about hotel Hilbert.

Example 3.3 Consider once again hotel Hilbert with infinitely many single rooms

$$Z(0), Z(1), Z(2), \ldots$$

that are all occupied by guests. Now, an infinite bus is arriving. This bus has infinitely many seats

$$B(0), B(1), B(2), \dots\ ,$$

and all seats are occupied by passengers[9]. The bus driver is asking the porter, whether he can accommodate all passengers. As usual, the porter answers: "No problem", and does the following:

He asks each guest from room $Z(i)$ to move to room $Z(2i)$ as depicted in the upper part of Fig. 3.18. After the move, each former guest has her or his own room and all rooms with odd numbers $1, 3, 5, 7, \dots, 2i + 1 \dots$ are empty. Now, it remains to match the free rooms with the bus passengers. The porter assigns room $Z(1)$ to the passenger sitting on seat $B(0)$, room $Z(3)$ to the passenger sitting on sit $B(1)$, etc. In general, the passenger from $B(i)$ gets room $Z(2i + 1)$ as depicted in Fig. 3.18. In this way, one gets the matching

$$(B(0), Z(1)), (B(1), Z(3)), (B(2), Z(5)), \dots, (B(i), Z(2i + 1)), \dots$$

between the empty rooms with odd numbers and the seats of the infinite bus.

□

Exercise 3.7 a) Hotel Hilbert is only partially occupied. All rooms $Z(0), Z(2), Z(4), \dots, Z(2i), \dots$ with even numbers are occupied and all rooms with odd numbers are free. Now, two infinite buses B_1 and B_2 are coming. The seats of the buses are numbered as follows:

$$B_1(0), B_1(1), B_1(2), B_1(3), \dots$$

$$B_2(0), B_2(1), B_2(2), B_2(3), \dots$$

How can the porter act in order to accommodate all guests? Is it possible to accommodate all newcomers without asking somebody to move to another room?

b) Hotel Hilbert is fully occupied. Now, three infinite buses are coming. The seats of each bus are enumerated by natural numbers. How can the porter accommodate everybody?

Exercise 3.8 Show by matching $\mathbb{Z}$ and $\mathbb{N}$ that

$$|\mathbb{Z}| = |\mathbb{N}|$$

holds, where $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ is the set of all integers.

[9] Each seat is occupied by exactly one passenger.

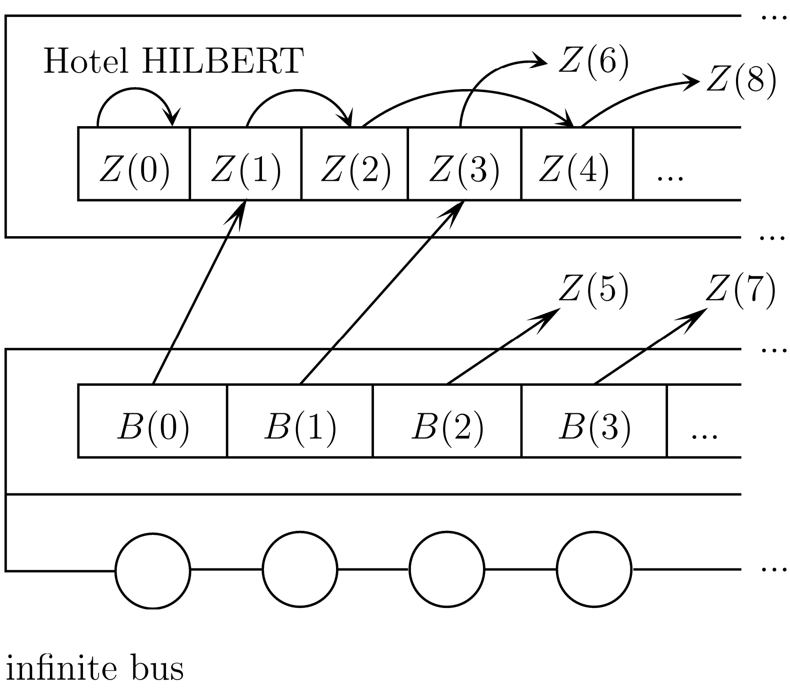

Fig. 3.18

Exercise 3.9 (challenge) Let $[a, b]$ be the set of all points (all real numbers) of the real axes between a and b.

a) Show that
$$|[0, 1]| = |[1, 10]| \ .$$
Try to show it by geometric means as in Example 3.2.

b) Prove
$$|[0, 1]| = |[1, 100]|$$
by arithmetic arguments, i.e., find a function f such that the pairs $(f(i), i)$ for $i \in [0, 100]$ build a matching of $[0, 1]$ and $[0, 100]$.

Exercise 3.10 (challenge) Assume that hotel Hilbert is empty, i.e., there is no guest accommodated in the hotel. Since all used accommodation strategies were based on moving former guests from a room to another, there is the risk that to stay in the hotel becomes unpopular. Therefore, the porter needs an accommodation strategy that does not require any move of an already accommodated guest. This accommodation strategy has to work even if arbitrarily many finite and infinite buses arrive in arbitrarily many different moments. Can you help the porter?

We observe that proving

$$|\mathbb{N}| = |A|$$

for a set A does not mean nothing else than numbering all elements of the set A by natural numbers. A matching between $\mathbb{N}$ and A unambiguously assigns a natural number from $\mathbb{N}$ to each element of A. And this assigned natural number can be viewed as the order of the corresponding element of A. For instance, if $(3, \text{John})$ is a pair of the matching, then John can be viewed as the third element of the set A. Vice versa, each numbering of elements of a set A directly provides a matching between $\mathbb{N}$ and A. The pair of the matching are simply

$$\text{order of } a, a$$

for each element a of A. In what follows, the notion of **numbering**[10] the elements of A enables to present transparent arguments for claims $|\mathbb{N}| = |A|$ for some sets A, i.e., for showing that A has as many elements as $\mathbb{N}$.

The matching

$$(0, 0), (1, 1), (2, -1), (3, 2), (4, -2), (5, 3), (6, -3), \ldots$$

of the sets $\mathbb{N}$ and $\mathbb{Z}$ assigns the following order to the elements of $\mathbb{Z}$:

$$0, 1, -1, 2, -2, 3, -3, \ldots \; .$$

In this way 0 is the 0-th element, 1 is the first element, -1 is the second element, 2 is the third element, etc.

Exercise 3.11 Assign to $\mathbb{Z}$ another order of elements than the presented above by giving another matching.

Exercise 3.12 Prove that

$$|\mathbb{N}| = |\mathbb{N}_{quad}| \; ,$$

where $\mathbb{N}_{quad} = \{i^2 \mid i \in \mathbb{N}\} = \{0, 1, 4, 9, 16, 25, \ldots\}$ is the set of all squares of natural numbers. Which order of the elements of $\mathbb{N}_{quad}$ do you get by the matching you proposed?

[10] In the scientific literature one usually uses the term “enumeration” of the set A.

Our attempt to answer the next question increases the hardness degree of our considerations. What is the relation between $|\mathbb{N}|$ and $|\mathbb{Q}^+|$? Remember that

$$\mathbb{Q}^+ = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}^+ \right\}$$

is the set of all positive rational numbers. We have already observed that calculating repeatedly averages one can show that there are infinitely many rational numbers between any two rational numbers a and b with $a < b$. If one partitions the real axes into infinitely many parts $[0, 1], [1, 2], [2, 3], \ldots$ as depicted in Fig. 3.19, then the cardinality of $\mathbb{Q}^+$ looks like

$$\infty \cdot \infty = \infty^2$$

because each of these infinitely many parts (intervals) contains infinitely many rational numbers.

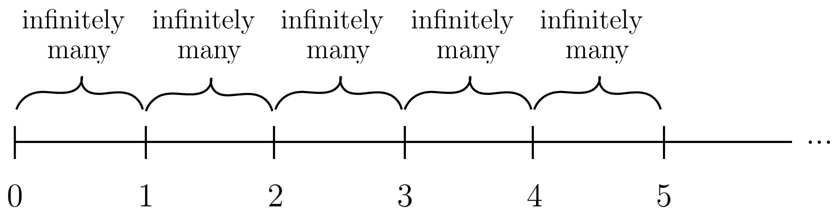

Fig. 3.19

At the first glance, trying to prove the equality $|\mathbb{N}| = |\mathbb{Q}^+|$ does not seem very promising. The natural numbers $0, 1, 2, 3, \ldots$ lay very thin on the right half of th the axes and between any two consecutive natural numbers i and $i + 1$ there are infinitely many rational numbers. Additionally, we know that a matching between $\mathbb{N}$ and $\mathbb{Q}^+$ would provide a numbering of element in $\mathbb{Q}^+$. How can such a numbering of positive rational numbers look like? It cannot follow the size of the rational numbers, because, as we know, there is no smallest positive rational number[11].

Though this very clear impression, we show the equality

$$|\mathbb{N}| = |\mathbb{Q}^+|$$

and so, in some sense that

$$\infty \cdot \infty = \infty$$

holds.

Observe first that the set $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ also does not have any smallest number, and though we can number their elements as follows:

$$0, -1, 1, -2, 2, -3, 3, \ldots\ .$$

The idea for $\mathbb{Q}^+$ is to write all positive rational numbers on an infinite sheet as follows (mathematicians among us would say that one assigns positions of the two-dimensional infinite matrix to positive rational numbers). Each positive rational number can be written as

$$\frac{p}{q}\ ,$$

where p and q are positive integers. We partition the infinite sheet of paper into infinitely many columns and infinitely many rows. We number the rows by

$$1, 2, 3, 4, 5, \ldots$$

from top to the bottom and we number the columns from the left to the right (Fig. 3.20). We place the fraction

$$\frac{i}{j}$$

on the square, in which the i-th row intersects the j-th column. In this way we get the situation (the infinite matrix) as described in Fig. 3.20.

[11] For any small rational number a, one can get the smaller rational number $a/2$ by halving a.

	1	2	3	4	5	6	...
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	...
2	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{6}$	...
3	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	...
4	$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$	$\frac{4}{6}$	...
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	$\frac{5}{6}$	...
6	$\frac{6}{1}$	$\frac{6}{2}$	$\frac{6}{3}$	$\frac{6}{4}$	$\frac{6}{5}$	$\frac{6}{6}$	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Fig. 3.20

We do not have any doubt that this infinite sheet (this infinite matrix) contains all positive fractions. If one looks for an arbitrary fraction p/q, one immediately knows that p/q is placed on the intersection of the p-th row and the q-th column. But we have another problem. Some[12] positive rational numbers occur on the sheet several times, even infinitely many times. For instance, the number 1 can be represented as a fraction in the following different ways:

$$\frac{1}{1}\,,\ \frac{2}{2}\,,\ \frac{3}{3}\,,\ \frac{4}{4}\,,\ \ldots\ .$$

The rational number $1/2$ can be written as

$$\frac{1}{2}\,,\ \frac{2}{4}\,,\ \frac{3}{6}\,,\ \frac{4}{8}\,,\ \ldots\ .$$

Exercise 3.13 Which infinitely many representations as a fraction does the rational number $\frac{3}{7}$ have?

But we aim to have each positive rational number exactly once on this sheet. Therefore, we take the fraction p/q that cannot be

[12]in fact all

reduced[13] as a unique representation of the rational number p/q. In this way 1 is uniquely representing $1/1$, one half is represented by $1/2$, because all other fractions represented by 1 and $1/2$ can be reduced. Hence, we remove (rub out) all fractions of the sheet that can be reduced. In this way we get empty positions (squares) on the intersections of some rows and columns, but this is not disturbing us.

	1	2	3	4	5	6	...
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	...
2	$\frac{2}{1}$		$\frac{2}{3}$		$\frac{2}{5}$		...
3	$\frac{3}{1}$	$\frac{3}{2}$		$\frac{3}{4}$	$\frac{3}{5}$		...
4	$\frac{4}{1}$		$\frac{4}{3}$		$\frac{4}{5}$		...
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$		$\frac{5}{6}$	...
6	$\frac{6}{1}$				$\frac{6}{5}$		...
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Fig. 3.21

Now we want to number the fractions in Fig. 3.21 as the first, the second, the third, etc. Clearly, we cannot do it in the way in which first the elements (fractions) of the first row are numbered, then the elements of the second row, etc. The reason for this impossibility is that the number of elements of the first row is infinite. We would fail in such an attempt because we would never start to number the elements of the second row. The first row would simply consume all numbers of $\mathbb{N}$. Analogously, it is impossible to number the elements of the infinite sheet column by column. What can we do then? We number the elements of the sheet in Fig. 3.21

[13] The greatest common divisor of p and q is 1.

diagonal by diagonal. The **k-th diagonal of the sheet** contains all positions (Fig. 3.22) for which the sum of its row number i and its column number j is $k+1$ ($i+j=k+1$).

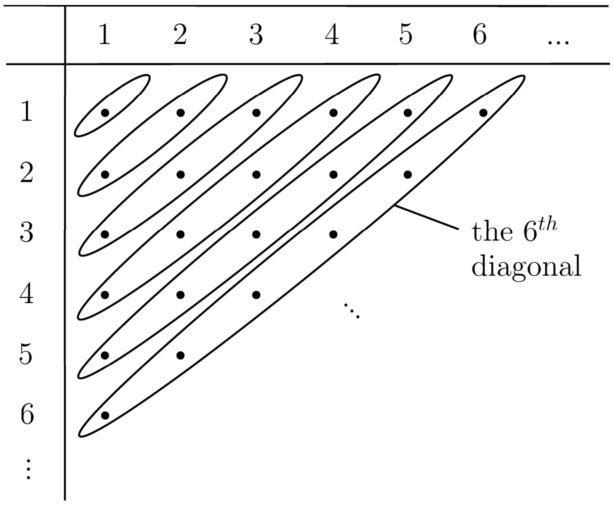

Fig. 3.22

In this way the first diagonal contains only one element $\frac{1}{1}$. The second diagonal contains two elements $\frac{2}{1}$ and $\frac{1}{2}$. And, for instance, the fourth diagonal contains the four elements $\frac{4}{1}, \frac{3}{2}, \frac{2}{3}$, and $\frac{1}{4}$. In general, for each positive integer k, the k-th diagonal contains exactly k positions and so at most k fractions.

Now, we order (number) the positions of the infinite sheet and in this way the fractions laying there in the way depicted in Fig. 3.23.

We order the diagonals according to their numbers and we order the elements of any diagonal from the left to the right. Following this strategy and the placement of the fractions in Fig. 3.21, we obtain the following numbering of all positive rational numbers:

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{1}{3}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{5}{1}, \frac{1}{5}, \frac{6}{1}, \frac{5}{2}, \frac{4}{3}, \frac{3}{4}, \frac{2}{5}, \frac{1}{6}, \ldots$$

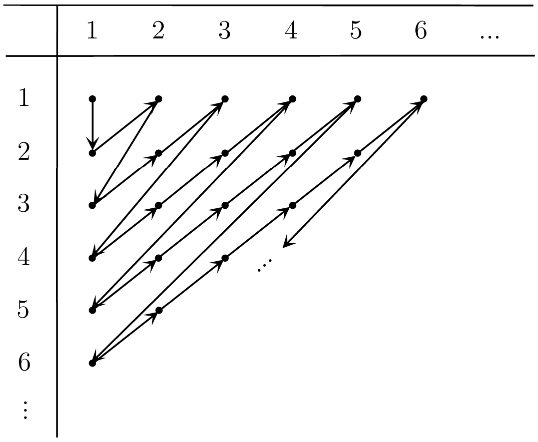

Fig. 3.23

Following our numbering convention, 1/1 is the 0-th rational number, 2/1 is the first positive rational number, etc. For instance, 3/1 is the third rational number and 5/2 is the 12-th one.

Exercise 3.14 Extend the matrix in Fig. 3.21 by two more rows and columns and place the corresponding fractions on their visible positions. Use this extended matrix to write the sequence of fractions that got the orders $17, 18, 19, \ldots, 26, 27$ by our numbering.

The most important observation for seeing the correctness of our numbering strategy is that each positive rational number (fraction) got assigned a natural number as its order. The argument is straightforward. Let p/q be an arbitrary positive fraction. The rational number p/q is placed on the intersection of the p-th row and the q-th column and so it lies on the diagonal $(p+q-1)$. Because **each diagonal contains finitely many positions (fractions)**, the numbering of elements of the forthcoming diagonals $1, 2, 3, \ldots, p+q-2$ is completed in a finite time and so the numbering of the elements of the diagonal $p+q-1$ will be performed too. In this way, p/q as an element of the diagonal $p+q-1$ gets its order, too. Since the i-th diagonal contains at most i rational numbers, the order of p/q is at most

$$1 + 2 + 3 + 4 + \ldots + (p + q - 1) \; .$$

In this way, one can conclude that

$$|\mathbb{Q}^+| = |\mathbb{N}|$$

holds.

Exercise 3.15 Fig. 3.24 shows another strategy for numbering of positive rational numbers that is also based on the consecutive numbering of diagonals. Write the first 20 positive rational numbers with respect to this numbering. Which order is assigned to the fraction 7/3? Which order does have the number 7/3 in our original numbering following the numbering strategy depicted in Fig. 3.23?

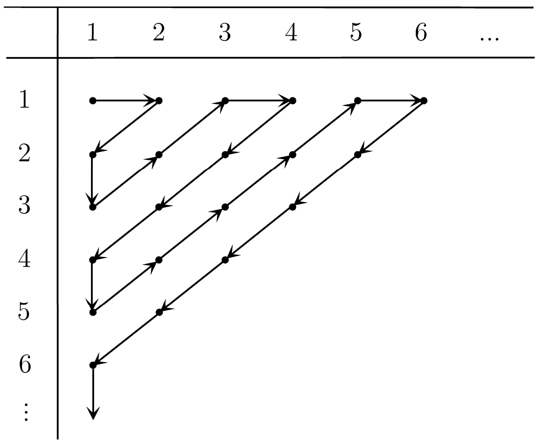

Fig. 3.24

Exercise 3.16 Hotel Hilbert is completely empty, no guest is living there. At once (as the real life sometimes brings), infinitely many infinite buses are arriving. The buses are numbered as

$$B_0, B_1, B_2, B_3, \ldots \; ,$$

i.e., there are as many as $|\mathbb{N}|$. For each $i \in \mathbb{N}$, bus B_i contains infinitely many seats

$$B_i(0), B_i(1), B_i(2), B_1(3), \ldots \; .$$

Each seat is occupied by exactly one passenger. How can the porter accommodate all passengers?

Exercise 3.17 (challenge) Prove that $|\mathbb{Q}| = |\mathbb{N}|$.

Exercise 3.18 (challenge) We define

$$\mathbb{N}^3 = \{(i, j, k) \mid i, j, k \in \mathbb{N}\}$$

as the set of all triples (i, j, k) of natural numbers. One can place any natural number on each of the three positions of a triple. Hence, one could say that $|\mathbb{N}^3| = |\mathbb{N}| \cdot |\mathbb{N}| \cdot |\mathbb{N}| = \infty \cdot \infty \cdot \infty = \infty^3$. Show that $|\mathbb{N}^3| = |\mathbb{N}|$, and so that $\infty = \infty^3$ holds.

3.3 There are Different Infinite Sizes or Why There are More Real Numbers Than Natural Ones

In Section 3.2 we learned the concept of Cantor for comparing the cardinalities of sets. Surprisingly, we discovered that the property distinguishing infinite objects from finite ones is that infinite objects contain proper parts that are as large as the whole. We were not successful in searching for an infinity that would be larger than $|\mathbb{N}| = \infty$. Even the unexpected equality $|\mathbb{Q}^+| = |\mathbb{N}|$ holds. And this is true though the rational numbers are infinitely denser placed on the axes than the natural ones. This means that $\infty \cdot \infty = \infty$. For each positive integer i, one can even prove that the infinite number

$$\underbrace{|\mathbb{N}| \cdot |\mathbb{N}| \cdot \ldots \cdot |\mathbb{N}|}_{k\ times} = \underbrace{\infty \cdot \infty \cdot \ldots \cdot \infty}_{k\ times} = \infty^k$$

is again the same as $|\mathbb{N}| = \infty$.

We are not far from believing that all infinite sets are of the same size. The next surprise is that the contrary is true. In what follows we show that

$$|\mathbb{R}^+| > |\mathbb{N}| .$$

Before reading Section 3.2 one would probably believe that the number of real numbers is greater than the number of natural numbers. But now we know that $|\mathbb{Q}^+| = |\mathbb{N}|$ holds. And the real numbers have similar properties as the rational numbers. There is no smallest positive real number and there are infinitely many real numbers on the real axes between any tow different real numbers. Since $|\mathbb{N}| = |\mathbb{Q}^+|$, the inequally $|\mathbb{R}^+| > |\mathbb{N}|$ would directly imply

$$|\mathbb{R}^+| > |\mathbb{Q}^+| \ .$$

Is this not surprising? Later in Chapter 4, we will get deeper understanding of the difference between the sets $\mathbb{R}$ and $\mathbb{Q}$ that is also responsible for the truthfulness of $|\mathbb{R}| > |\mathbb{Q}|$. At this place, we reveal only the idea that, in contrast to real numbers, all rational numbers have a finite representation as fractions. Most of the real numbers do not posses any finite description. In order to prove $|\mathbb{R}^+| > |\mathbb{N}|$, we even prove a stronger result. Let $[0, 1]$ be the set of all real numbers between 0 and 1, the numbers 0 and 1 included. We show

$$|[0, 1]| \neq |\mathbb{N}| \ .$$

How can one prove inequality between the cardinalities (sizes) of two infinite sets? For proving equality, one has to find a matching between the two sets considered. This can be complicated, but in some sense it is easy because of its constructive. You find a matching and the work is done. For proving $|A| \neq |B|$ you have to prove that **there does not exist any matching between *A* and *B***.The problem is that there may exist infinitely many strategies for constructing a matching between A and B. How to exclude the success of any of theses strategies? You cannot check all these infinitely many approaches one after another. When one has to show that something does not exist, then we speak about **proofs of nonexistence**.

> *To prove the nonexistence of an object or the impossibility of an event is the hardest task one can pose to a researcher in natural sciences.*

The word "impossible" is almost forbidden and if one uses it, then we have to take care on its exact interpretation. A known physician told me that there is possible to reconstruct the original egg from an egg fried in the pan. All is based on the reversibility of physical processes[14] and he was even able to calculate the prob-

[14] as formulated by quantum mechanics

ability of successing in the attempt of creating the original egg. The probability was so small that one could consider the success as a real miracle, but it was greater than 0. There are many things considered to be impossible, and though they are possible.

In mathematics we work in an artificial world and because of that we are able to create many proofs of nonexistence of mathematical objects. What remains is the fact that proof of nonexistence belong to the hardest argumentations in mathematics, too.

Let us try to prove that it is impossible to number all real numbers from the interval $[0,1]$, and so that $|[0,1]| \neq |\mathbb{N}|$. As already mentioned, we do it by indirect argumentation. We assume that there is a numbering of real numbers from $[0,1]$, and then we show that this assumption leads to a contradiction, i.e., that a consequence of this assumption is an evident nonsense[15].

If there is a numbering of real numbers in $[0,1]$ (i.e., if there is a matching between $[0,1]$ and $\mathbb{N}$), then one can make a list of all real numbers from $[0,1]$ in a table as depicted in Fig. 3.25.

This means that the first number in the list is

$$0.a_{11}a_{12}a_{13}a_{14}\ldots\ .$$

The symbols $a_{11}, a_{12}, a_{13}, \ldots$ are digits. In this representation, a_{11} is the first digit behind the decimal point, a_{12} is the second digit, a_{13} is the third one, etc. In general

$$0.a_{i1}a_{i2}a_{i3}a_{i4}\ldots$$

is the i-th real number from $[0,1]$ in our list (numbering). Our table is infinite in both directions. The number of rows is $|\mathbb{N}|$ and the number of columns is also $|\mathbb{N}|$, where the j-th column contains j-th digits behind the decimal points of all numbered real numbers in the list. The number of columns must be infinite, because most

[15] Here, we recommend to have a look at the schema of indirect proofs presented in Chapter 1. If a consequence of an assertion Z is a nonsense or contradicts something known, then the indirect proof schema says that Z does not hold, i.e., that the contrary of Z holds. The contrary of the existence of a matching between $[0,1]$ and $\mathbb{N}$ is the nonexistence of any matching between $[0,1]$ and $\mathbb{N}$.

	0	1	2	3	4	$\dots$	i	$\dots$
1	0.	$\boxed{a_{11}}$	a_{12}	a_{13}	a_{14}	$\dots$	a_{1i}	$\dots$
2	0.	a_{21}	$\boxed{a_{22}}$	a_{23}	a_{24}	$\dots$	a_{2i}	$\dots$
3	0.	a_{31}	a_{32}	$\boxed{a_{33}}$	a_{34}	$\dots$	a_{3i}	$\dots$
4	0.	a_{41}	a_{42}	a_{43}	$\boxed{a_{44}}$	$\dots$	a_{4i}	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$		
i	0.	a_{i1}	a_{i2}	a_{i3}	a_{i4}	$\dots$	a_{ii}	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Fig. 3.25

real numbers cannot be exactly represented by a bounded number of decimal positions behind the point. For instance, already the representation of the fraction

$$\frac{1}{3} = 0.\overline{3} = 0.33333\dots$$

requires infinitely many digits behind the decimal point. On the other hand this real number is nice because it is periodic. Numbers such as $\sqrt{2}/2$ and $\pi/4$ are not periodic and require infinitely many positions behind the decimal point for their decimal representation.

To be more transparent, we depict a concrete fraction of a hypothetical list of all real numbers from $[0, 1]$ in Fig. 3.26 by exchanging the abstract symbols a_{ij} for concrete digits.

In this hypothetical list the number $0.732110\dots$ is the first real number, $0.000000\dots$ is the second real number, etc.

In what follows, we apply the so-called **diagonalization method** in order to show that there is a real number from $[0, 1]$ missing

	0	1	2	3	4	5	6	...
1	0.	[7]	3	2	1	1	0	...
2	0.	0	[0]	0	0	0	0	...
3	0.	9	9	[8]	1	0	3	...
4	0.	2	3	4	[0]	7	8	...
5	0.	3	5	0	1	[1]	2	...
6	0.	3	1	4	0	5	[7]	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱
i	0.	7	6	5	0	0	1	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Fig. 3.26

in the list (Fig. 3.25). This contradicts our assumption that one has a numbering of the elements of $[0, 1]$ (i.e., each number from $[0, 1]$ has to occur in the list exactly once). Hence, our hypothetical numbering is not a numbering and we are allowed to conclude that there does not exist any numbering of the elements from $[0, 1]$.

Next, we construct a number c from $[0, 1]$ that is not represented by any row of the table (list), i.e., that differs from all numbers of the list. We create c digit by digit. We write c as

$$c = 0.c_1c_2c_3c_4 \dots c_i \dots \; ,$$

i.e., c_i is the i-th digit of c behind the decimal point. We choose $c_1 = a_{11} - 1$ if $a_{11} \neq 0$, and we set $c_1 = 1$ if $a_{11} = 0$. For the hypothetical numbering in Fig. 3.26 this means that $c_1 = 6$ because $a_{11} = 7$. Now we know with certaincy that c is different from the number written in the first row of our list in Fig. 3.25 (Fig. 3.26). The second digit c_2 of c is again chosen in such a way that it differs from a_{22}. We take $c_2 = a_{22} - 1$ if $a_{22} \neq 0$, and we set $c_2 = 1$ if $a_{22} = 0$. Hence, c differs from the number in the second row of the list, and so c is not the second number of the hypothetical numbering. Next, one chooses c_3 in such a way that $c_3 \neq a_{33}$ in order to assure that c is not represented by the third row of the list.

In general, one chooses $c_i = a_{ii} - 1$ for $a_{ii} \neq 0$ and $c_i = 1$ for $a_{11} = 0$. In this way c differs from the i-th number of our hypothetical numbering. After 6 construction steps for the table in Fig. 3.26 one gets

$$0.617106\ldots .$$

We immediately see that c differs from the numbers in the first 6 rows of the table in Fig. 3.26.

We observe that c differs from each number of the list at least in one decimal digit and so c is not in the list. Therefore, the table in Fig. 3.26 is not a numbering of $[0, 1]$. A numbering of $[0, 1]$ has to list all real numbers from $[0, 1]$ and c is clearly in $[0, 1]$. Hence, our assumption that one has a numbering of $[0, 1]$ (that there exists a numbering of $[0, 1]$) is false. We are allowed to conclude

> *"There does not exist any numbering of* $[0, 1]$*, and so there is no matching between* $\mathbb{N}$ *and* $[0, 1]$*"*

Exercise 3.19 Draw a table (similarly as we did in Fig. 3.26) of a hypothetic numbering of $[0, 1]$ that starts with the numbers $1/4, 1/8, \sqrt{2}/2, 0, 1, \pi/4, 3/7$. Use this table to determine the digits $c_1, c_2, \ldots, c_7$ of the number c in such a way that c differs from the numbers in the first 7 rows of your table.

Exercise 3.20 Consider a hypothetical numbering of $[0, 1]$, such that the 100-th number is $2/3$. Which digit of c is determined by this information?

Exercise 3.21 Determine the first 7 digits of c behind the decimal point of a hypothetic numbering of $[0, 1]$ presented in Fig. 3.27.

What exactly did we show and what was our argumentation? Assume, somebody says: "I have a numbering of $[0, 1]$." We discovered a method, called diagonalization, that enables us to reject any proposal of numbering of $[0, 1]$ as incomplete because at least one number from $[0, 1]$ is missing there. Since we can do it for each

	0	1	2	3	4	5	6	7	...
1	0.	**2**	0	0	1	7	8	0	...
2	0.	1	**7**	3	1	7	8	4	...
3	0.	1	6	**4**	3	3	3	3	...
4	0.	1	6	3	**0**	7	8	4	...
5	0.	1	6	3	1	**8**	8	4	...
6	0.	1	6	3	1	7	**9**	4	...
7	0.	1	6	3	1	7	8	**4**	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Fig. 3.27

hypothetic numbering of the elements of $[0, 1]$, there does not exist any (complete) numbering of $[0, 1]$.

Another point of view is that of indirect argumentation introduced in Chapter 1. Our aim was to prove the claim Z that there does not exist any numbering of $[0, 1]$. We start with the opposite claim $\overline{Z}$ and show that a consequence of $\overline{Z}$ is a nonsense. In this moment we reached our goal. The assection $\overline{Z}$ as the opposite of Z is the claim that there exists a numbering of the elements of $[0, 1]$. Starting from $\overline{Z}$ we show that in any such numbering of $[0, 1]$ one number from $[0, 1]$ is missing. This is a nonsense because no number is allowed to be missing in a numbering. Therefore, $\overline{Z}$ does not hold and so there does not exist any numbering of $[0, 1]$.

Since we cannot number the elements of $[0, 1]$ (there is no matching between $\mathbb{N}$ and $[0, 1]$), we cannot number the elements of $\mathbb{R}^+$, too.

Exercise 3.22 Explain, why the nonexistence of a numbering of the elements of $[0, 1]$ implies the nonexistence of a numbering of the elements of $\mathbb{R}^+$.

[Hint: You can try to explain how to transform each numbering of $\mathbb{R}^+$ into a numbering of $[0, 1]$. Why is this a correct argument?]

Since $\mathbb{N} \subset \mathbb{R}^+$ and there is no matching between $\mathbb{N}$ and $\mathbb{R}^+$, we can conclude that

$$|\mathbb{N}| < |\mathbb{R}^+|$$

holds. Hence, there are at least two infinite sets of different sizes, namely $\mathbb{N}$ and $\mathbb{R}^+$. One can even show that there are unboundedly many (infinitely many) different infinite sizes. We omit to deal with the technical proof of this result here because we do not need it for reaching our main goal. We are ready to show in the next chapter that the number of computing tasks is larger than the number of algorithms, and so that there exist problems that cannot be algorithmically (automatically by the means of computers) solved.

Exercise 3.23 Let us change the diagonalization method presented in Fig. 3.25 a little bit. For each $i \in \mathbb{N}$, we choose $c_i = a_{i,2i} - 1$ for $a_{i,2i} \neq 0$ and $c_i = 1$ for $a_{i,2i} = 0$.

a) Are we allowed again to say that the number $0.c_1c_2c_3c_4\ldots$ is not included in the list? Argue for your answer!
b) Frame the digits $a_{i,2i}$ of the table in Fig. 3.25.
c) Which values are assigned to c_1, c_2, and c_3 for the hypothetic list in Fig. 3.27 in this way? Explain, why the created number $c = 0.c_1c_2c_3\ldots$ is not among the first three numbers of the table.

3.4 The Most Important Ideas Once Again

Two infinite sizes can be compared. One has to represent them by the cardinalities of the two sets. Using this base, Cantor introduced the concept for comparing infinite sizes (cardinalities) of two sets by the shepherd principle. Two sets are equally sized, if one can match their elements. A set A has the same cardinality as $\mathbb{N}$, when one can number all elements of A by natural numbers. Clearly, each numbering of A corresponds to a matching between A and $\mathbb{N}$. Surprisingly, one can match $\mathbb{N}$ and $\mathbb{Z}$, though $\mathbb{N}$ is a proper part of $\mathbb{Z}$. In this way we recognized that the property

"to have a proper part that is as large as the whole"

is exactly the characteristics that enables to distinguish finite objects from infinite ones. No finite object may have this property. For infinite objects, this is a must. Though there are infinitely many rational numbers between any two consecutive natural numbers i and $i + 1$, we found a clever enumeration[16] of all positive rational numbers and so we showed that $|\mathbb{N}| = |\mathbb{Q}^+|$. After that, we applied the schema of indirect proofs in order to show that there is no numbering of all positive real numbers, and so that there is no matching between $\mathbb{N}$ and $\mathbb{R}^+$.

In Chapter 4, it remains to show that the number of programs is equal to $|\mathbb{N}|$ and that the number of algorithmic tasks is at least $|\mathbb{R}^+|$.

In Chapter 3, we did not present any miracle of computer science. But we investigated the nature of infinity and the concept of comparing infinite sizes, and in this way we learned miracles of mathematics that are real jewels of the science fundamentals. Jewels rarely lay on the street and one has usually to do something to obtain them. Therefore, we also required to sweat a little bit in order to grasp infinity. And so, one may not be surprised that walking around our path to the computer science miracles can be strenuous. But tenacity is a good property and the aim is worth the effort. Let us stay the next two chapters and then we will witness one miracle after the other. We will experience unexpected and elegant solutions to hopeless situations that increases the pulse of each friend of science. Only by patience and a hard work, one can attain knowledge that is really valuable.

Solutions to Some Exercices

Exercise 3.1 For the sets $A = \{2, 3, 4, 5\}$ and $B = \{2, 5, 7, 11\}$ there are $4! = 24$ different matchings. For instance,

$$(2, 11), (3, 2), (4, 5), (5, 7)$$

or

$$(2, 11), (3, 7), (4, 5), (5, 2) \ .$$

[16] not according to their sizes