The sequence of pairs $(2,2),(4,5),(5,11),(2,7)$ is not a matching between A and B because element 2 of A occurs in two pairs $(2,2)$ and $(2,7)$ and element 3 of A does not occur in any pair.

Exercise 3.8 A matching between $\mathbb{N}$ and $\mathbb{Z}$ can be found in such a way that one orders the elements of $\mathbb{Z}$ in the following sequence

$$0,1,-1,2,-2,3,-3,4,-4,\ldots,i,-i,\ldots$$

and then creates a matching by assigning to each element of 7 its order in this sequence. In this way we get the matching

$$(0,0),(1,1),(2,-1),(3,2),(4,-2),(5,3),(6,-3),\ldots\ .$$

In general we build the pairs

$$(0,0),(2i,-i)\text{ and }(2i-1,i)$$

for all positive integers i.

Exercise 3.10 (challenge) First, the porter partitions all rooms into infinitely many groups, each of an infinite size. Always when a group of guests arrived (does not matter whether the group is finite or infinite), the porter accommodates the guest in the next (still unused) group of rooms.

As usual for the staff of Hilbert hotel, the porter is well educated in mathematics and so he knows that there are infinitely many primes

$$2,3,5,7,11,13,17,19,\ldots\ .$$

Let p_i be the i-th prime of this sequence. The porter uses p_i to determine the i-th infinite group of natural numbers as follows:

$$\text{group}(i)=\{p_i,p_i^2,p_i^3,p_i^4,\ldots,(p_i)^j,\ldots\}\ .$$

For instance, group(2)$=\{3,9,27,81,\ldots\}$. Due to the knowledge of the fundamental theorem of arithmetics, the porter knows that no natural number belongs to more than one group. Using this partition of rooms into the groups with respect to their room numbers, the porter can assign the rooms to the guests without any more even when infinitely many groups of guests arrive one after each other. It does not matter, whether the i-th group of guest is finite or infinite, the porter books the whole room group group(i) for the i-th guest group. If the guest of the i-th group are denoted as

$$G_{i,1},G_{i,2},G_{i,3},\ldots,G_{i,j},\ldots$$

then guest $G_{i,1}$ gets the room $Z(p_i)$, guest $G_{i,2}$ gets room $Z(p_i^2)$, etc.

Exercise 3.12 The sequence of pairs

$$(0,0),(1,1),(2,4),(3,9),(4,16),\ldots,(i,i^2),\ldots$$

is a matching between $\mathbb{N}$ and $\mathbb{N}_{quad}$. We see that each number from $\mathbb{N}$ appears exactly once as the first element in a pair, and analogously each integer from $\mathbb{N}_{quad}$ can be found exactly once as the second element of a pair.

Exercise 3.20 The decimal representation of the fraction 2/3 is

$$0.\overline{6} = 0.666666\ldots\ .$$

Hence, the 100-th position behind the decimal point is also 6. Therefore, one sets $c_{100} = 6 - 1 = 5$.

Exercise 3.21 For the hypothetical numbering of real numbers form $[0, 1]$ in Fig. 3.27, one gets

$$c = 0.1631783\ldots\ .$$

Aufgabe 3.22 We perform an indirect proof by following the schema of the indirect argumentation from Chapter 1. We know that there is no numbering of $[0, 1]$. The aim is to show that there does not exist any numbering of $\mathbb{R}^+$. Assume the contrary of our aim, i.e., that there is a numbering of $\mathbb{R}^+$. We consider this numbering of $\mathbb{R}^+$ as a list and erase those numbers of this list that are not from $[0, 1]$. What remains is the list of numbers from $[0, 1]$ that is (without any doubts) a numbering of $[0, 1]$. But we know that there does not exist any numbering of $[0, 1]$ and so the contrary of our assumption must hold. The contrary of our assumption is our aim, i.e., that there does not exist any numbering of $\mathbb{R}^+$.