Exam for Randomized Algorithms Course

Tuesday, November 26, 2002, 9:15 - 12:00, IFW B42, written exam; material: lectures, exercises, reading assignments; books, notes, etc. allowed, (but no electronic devices). *It is understood that answers have to be substantiated!* That is, you have to provide proofs for the given answers.

Problem 1 *For* $x = (x_1, \ldots, x_n)$ *drawn uniformly at random from* $\{0,1\}^n$*, we have that for all pairs* $\{i,j\} \in \binom{\{1\ldots n\}}{2}$ *of indices,*

$$\Pr(x_i = x_j) = 1/2. \tag{1}$$

Provide a small sample space that satisfies (1) for all pairs $\{i, i+1\}$ *of successive indices. Next, provide a small sample space which satisfies (1) for all pairs* $\{i,j\}$*.*

Problem 2 *Let* $G = (V,E)$ *be a (simple undirected) graph with* k *cycles of length* 4 *(these cycles may overlap). Now we assign* 0*'s and* 1*'s to the vertices. A* 4*-cycle* alternates *under this assignment, if* 0*'s and* 1*'s alternate as we run around the cycle. Show that there is a constant* $c \in \mathbf{R}^+$ *such that there is always an assigment with* $c \cdot k$ *cycles of length* 4 *that alternate. Try to get* c *as large as you can. How many can we get, if* G *is the complete graph on* n *vertices,* $n \in \mathbf{N}$*?*

Problem 3 *Consider the following decision problem: Given a* $0/1$*-sequence*

$$x = (x_1, x_2, \ldots, x_n) \in \{0,1\}^n$$

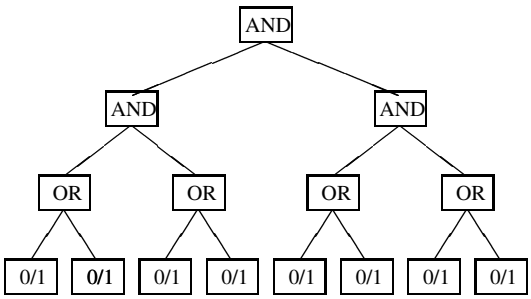
of odd length $n = 2k+1$*,* $k \in \mathbf{N}_0$*, decide whether* x *contains more 0's or more 1's by querying positions* x_i*,* $i \in \{1..n\}$*. Give a lower bound on the worst-case expected running time (i.e. number of positions to be queried) of any randomized algorithm which decides this problem.*

Problem 4 *Let* $G = (V,E)$ *be a (simple, undirected) graph. A* 4-cut *for* G *is a partition* $V = A_1 \cup A_2 \cup A_3 \cup A_4$*,* $A_i \cap A_j = \emptyset$ *for* $i \neq j$*, of the vertex set into four parts. The size of such a 4-cut is the number*

$$\#\{e \in E : e = \{v,w\}, v \in A_i \text{ and } w \in A_j \text{ for some } i \neq j\}$$

of edges between different parts. Give a randomized algorithm for finding large 4-cuts in graphs and analyze its performance (i.e., the expected size of the 4-cut that it produces). Derandomize the algorithm (for instance, by means of small sample spaces).

Problem 5 *Consider an AND-AND-OR-tree* T *of height 3.*



Design a randomized algorithm for evaluating (the root of) T *for given* $0/1$*-assignments to the leaves, and analyze its worst-case expected running time (i.e. derive bounds for its worst case expected running time).*

Problem 6 *Imagine a complete ternary tree* T *of height* h *(this means that every internal node has* 3 *children and every leaf has depth* h*). The nodes of* T *are colored in such a way that every internal node* v *has a child* w *of the same color as* v*. Suppose you can obtain information about the coloring only by asking questions of the form*

- *given a leaf* ℓ*, what is its color? (leaf evaluations)*
- *given an internal node* v *and a child* w whose color is known*, do* v *and* w *have the same color? (edge evaluations).*

Then the color of the root can trivially be computed by evaluating all 3^h *leaves and all* $\sum_{i=1}^{h} 3^i = (3^{h+1} - 3)/2$ *edges. Develop and analyze a randomized algorithm which finds the color of the root with an expected number of* $O(2^h)$ *leaf and edge evaluations.*

Problem 7 *Let* $G = (V, E)$ *be a bipartite (simple, undirected) graph, such that there exists* $k \in \mathbf{N}$ *with* $\deg_G v = k \cdot i_v$*,* $i_v \in \mathbf{N}_0$*, for all* $v \in V$*. (That is, all node degrees are integer multiples of* k*.) Show that there is always a subgraph* $G' = (V, E')$ *of* G *(i.e.* $E' \subseteq E$*) such that* $\deg_{G'} v = i_v$ *for all* $v \in V$*, and design a randomized algorithm for finding such a subgraph. (Hint: Note that we have handled the special case of "*$i_v = 1$ *for all* $v \in V$*" and adapt the algorithm presented there for the more general situation here. Verify and analyze the adapted algorithm.)*