Special Exercise Set 4

December 1, 2009

due on Wednesday, December 16, in the lecture

- There will be a total of four special exercise sets during this semester.
- You are expected to solve them carefully and then write a nice and complete exposition of your solutions using LaTeX.
- You are welcome to discuss the tasks with your colleagues, but we expect each of you to hand in your own, individual writeup.
- Your solutions will be graded. The three highest out of your four achieved grades will account for 10% of your final grade for the course each (so 30% of the grade in total).
- We encourage you to work on "bonus exercises", but they are not required to achieve the best grade.

4.1 Antichains in the Cube

Consider the Hamming cube $\{0,1\}^n$. We define a partial ordering $\leq$ on $\{0,1\}^n$: For $x, y \in \{0,1\}^n$, we say $x \leq y$ if $x_i \leq y_i$ holds for all $1 \leq i \leq n$. An *antichain* is a set $S \subseteq \{0,1\}^n$ such that for all $x, y \in S$ with $x \neq y$, neither $x \leq y$ nor $y \leq x$ hold. Let $|x|_1$ denote the number of 1s in x, also called the *Hamming weight*. For any $1 \leq k \leq n$, the set $L_k := \{x \in \{0,1\} \mid |x|_1 = k\}$ is an antichain.

Exercise. Show that if S is an antichain, then $|S| \leq |L_{\lceil \frac{n}{2} \rceil}|$.

Hint: let $k := \min\{|x|_1 \mid x \in S\}$. Show that if $k \neq \left\lceil \frac{n}{2} \right\rceil$, you can replace the elements of Hamming weight k by elements of Hamming weight $k+1$, in a way that ensures that S is still antichain and does not become smaller.

4. 2 Matchings in the Cube

The n-cube $\{0,1\}^n$ consists of several levels $L_k := \{x \in \{0,1\} \mid |x|_1 = k\}$. For two consecutive levels L_{k-1}, L_k (here $1 \leq k \leq n$), the graph structure on $\{0,1\}^n$ induces a bipartite graph on $L_{k-1} \cup L_k$: the vertices $x \in L_{k-1}$ and $y \in L_k$ are connected by an edge if $d_H(x, y) = 1$.

Exercise. Show that if $k \leq \left\lceil \frac{n}{2} \right\rceil$, then the graph described above contains a matching of cardinality $|L_{k-1}|$, i.e. a matching saturating the smaller part of the bipartite graph.

4.3 The PPZ algorithm

Let V be a set of n variables and let F be a 3-CNF formula over V. Suppose $\alpha \in \{0,1\}^V$ is an isolated satisfying assignment of F. We have shown that the ppz algorithm returns α with probability at least $2^{-2n/3}$.

For a CNF formula F, a satisfying assignment x and a variable x, we call a clause C of F critical with respect to α and x if α satisfies exactly one literal of C, and this literal is a literal over x. Clearly, if α is isolated, then for every variable $x \in V$ there exists a clause $C \in F$ that is critical with respect to α and x.

Exercise a) Suppose somebody tells you that for every variable $x \in V$, there are *at least two* clauses $C, D \in F$ that are critical with respect to α and x. Prove a better success probability for ppz, i.e., show that there exists a constant $c < \frac{2}{3}$ such that ppz returns α with probability at least 2^{-cn}.

Exercise b) Let F be a 3-CNF formula over V and α an isolated satisfying assignment. In a first step, we "mark" every variable of V with probability p, independently. This gives us a set $U \subseteq V$. In a second step, we set each marked variable to its value under α. Formally, let $\beta = \alpha_{|U}$ and define $F' := F^{[\beta]}$. We say a variable $x \in V$ is *forced* if $x \in V$, and either $\{x\} \in F'$ or $\{\bar{x}\} \in F'$. Give a lower bound on the expected number of forced variables in terms of n and p and provide an example CNF formula that shows that your bound is best possible, for any n and p.