Satisfiability of Boolean Formulas

Candidate:

First name: ........................................................................................................

Last name: .......................................................................................................... 

Student ID (Legi) Nr.: ........................................................................................

I attest with my signature that I was able to take the exam under regular conditions and that I have read and understood the general remarks below.

Signature: ...........................................................................................................

General remarks and instructions:

1. You can solve the 4 exercises in any order. They are not ordered by difficulty or in any other meaningful way.
   
   For each exercise, the subtasks can be solved independently. The corresponding number of points of each subtask is indicated. Not all points are might be equally hard to get.

2. Check your exam documents for completeness (2 cover pages and 2 pages containing 4 exercises).

3. Immediately inform an assistant in case you are not able to take the exam under regular conditions. Later complaints are not accepted.

4. Pencils are not allowed. Pencil-written solutions will not be reviewed.

5. The exam is open-book. All written material is allowed but no electronics of whatever sort.

6. Attempts to cheat/defraud lead to immediate exclusion from the exam and can have judicial consequences.

7. Provide only one solution to each exercise. Cancel invalid solutions clearly.

8. All solutions must be understandable and well-founded. Write down the important thoughts in clear sentences and keywords. No points will be awarded for unfounded or incomprehensible solutions (except in the multiple-choice parts). You can write your solution in English or German.

9. You do not need to reprove (but can reference) things that were already proved in the lecture or in any of the exercises treated in class. But if you want to prove something different then you must point out all details that need to be done differently in your proof.

10. Make sure to write your student-ID (Legi-number) on all the sheets (and your name only on this cover sheet).

Good luck!
<table>
<thead>
<tr>
<th>achieved points (maximum)</th>
<th>reviewer’s signature</th>
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Exercise 1 - Multiple Choice (30 Pts)

Consider the following 5 claims and mark the corresponding boxes. Grading: 3 points for a correct marking without a correct justification, 6 points for a correct marking with a correct short justification, and -3 points for a wrongly marked box (you will receive non-negative total points in any case).

Note: You may write your justification on an additional sheet if you need more space. But do not spend too much time on lengthy arguments; keywords and rough sketches are sufficient here.

(a) Let \( F \) be an \((\geq 1)\)-CNF formula over \( V \) such that all 1-clauses of \( F \) consist of a positive literal.

Then there is assignment on \( V \) that satisfies at least a \( \frac{2}{3} \)-fraction of clauses of \( F \).

\[
\begin{array}{c}
\boxed{\text{True}} \quad \boxed{\text{False}} \\
\end{array}
\]

Justification: .................................................................

(b) Let \( n \geq 100 \), and let \( V \) be a set of \( n \) variables. Over \( V \), there are strictly more \( n \)-clauses than \((n-1)\)-clauses.

\[
\begin{array}{c}
\boxed{\text{True}} \quad \boxed{\text{False}} \\
\end{array}
\]

Justification: .................................................................

(c) Let \( F \) be a CNF formula over \( V \) and let \( D \geq 1 \). Consider \( \text{ppz}(F,V) \) and \( \text{ppsz}(F,V,D) \) (in the lecture notes \( \text{ppsz}(F,V,\{\},D) \)). For any satisfying assignment \( \alpha \) on \( V \) of \( F \), \( \text{ppsz} \) returns \( \alpha \) with at least the probability that \( \text{ppz} \) returns \( \alpha \).

\[
\begin{array}{c}
\boxed{\text{False}} \quad \boxed{\text{True}} \\
\end{array}
\]

Justification: .................................................................

(d) Suppose there are numbers \( c_1, c_2, \ldots, c_m \in \mathbb{N}_0 \) such that \( \sum_{i=1}^{m} 2^{-c_i} = 1 \). Then there exists an unsatisfiable CNF formula \( F \) with \( m \) clauses where the \( i \)-th clause has size \( c_i \).

\[
\begin{array}{c}
\boxed{\text{False}} \quad \boxed{\text{True}} \\
\end{array}
\]

Justification: .................................................................

(e) There is an \((\leq 3)\)-CNF formula \( F \) over 10 variables \( V \) with the property that an assignment \( \alpha \) on \( V \) satisfies \( F \) if and only if \( \alpha \) sets \textbf{at most} 3 variables to 1.

\[
\begin{array}{c}
\boxed{\text{True}} \quad \boxed{\text{False}} \\
\end{array}
\]

Justification: .................................................................
Exercise 2 - Many Satisfying Assignments (30 [6+6+6+6+6] Pts)

For a CNF formula $F$ over $V$ define $\text{ratio}(F, V) := \frac{\text{sat}(F)}{2^{|V|}}$. Let $\text{ratio}(F) := \text{ratio}(F, \text{vbl}(F))$.

Let $F$ be a $(\leq 3)$-CNF formula over $n \geq 1$ variables $V$.

(a) Show that $\text{ratio}(F) = \text{ratio}(F, V)$.

(b) Show that if $F \neq \emptyset$, there exists an assignment $\beta$ on at most 3 variables such that $\text{ratio}(F[\beta]) = 0$.

(c) Let $W \subseteq V$. Show that $\text{ratio}(F) = \frac{1}{2^n \cdot n} \sum_{\beta \in \{0,1\}^W} \text{ratio}(F[\beta])$.

(d) Suppose that $\text{ratio}(F, V) \geq \frac{1}{n}$.
   - Show that $F$ has a (non-total) satisfying assignment $\alpha$ on $O(\log n)$ variables (i.e. $F[\alpha] = \emptyset$).
   - Show that in addition to (d), such a satisfying assignment can be found in deterministic polynomial time.

Exercise 3 - Partial Satisfaction (30 [7+7+16] Pts)

(a) Let $F$ be a CNF formula with $2m$ clauses, consisting of $m$ 2-clauses and $m$ 3-clauses.
   - Show there exists an assignment satisfying at least a $\frac{11}{16}$-fraction of the clauses of $F$.

(b) Show that in addition to (a), such an assignment can be found in deterministic polynomial time.

(c) Let $F$ be a satisfiable $(\geq 2)$-CNF formula. Give a deterministic polynomial time algorithm that, given $F$, finds an assignment satisfying at least a $(\frac{3}{4} + \epsilon)$-fraction of the clauses of $F$ for some $\epsilon > 0$.

Exercise 4 - Algorithms for 3-Coloring (30 [10+10+10] Pts)

Let $G = (V, E)$ be a graph on $n$ vertices $V$. Remember that a proper 3-coloring of the vertices of $V$ is a mapping $\chi : V \to \{1, 2, 3\}$ such that $\chi(u) \neq \chi(v)$ whenever $\{u, v\} \in E$.

The 3-coloring problem is the decision problem of whether a given graph $G$ has a proper 3-coloring.

For two 3-colorings $\chi, \chi^*$ of $V$, we define the Hamming distance $d_H(\chi, \chi^*) := |\{v \in V | \chi(v) \neq \chi^*(v)\}|$, the number of vertices that are colored differently in $\chi$ and $\chi^*$.

(a) Give a constant size set of 3-colorings $C$ on $V$ such that for every possible 3-coloring $\chi$ there exists $\chi' \in C$ with $d_H(\chi, \chi') \leq \frac{1}{4} n$ (i.e. a covering code of radius at most $\frac{1}{4} n$).

(b) Adapt the searchball procedure (on page 141) to give a deterministic algorithm for the 3-coloring problem running in time $O(2^\frac{\Theta n}{n} \text{poly}(n))$.

(c) Adapt Schöning’s algorithm ("sch" on page 147) to find a proper 3-coloring. State where the algorithm needs to be changed, argue about the transition probabilities of the corresponding Markov chain, give the success probability of a single iteration and argue about the running time.
   - You do not need to restate the proof in the lecture notes, only the necessary changes.