Today's exercises

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5.1: Euler-Poincaré

Q: How many faces of dimension $i$ are there?

A: $\binom{n}{i} \cdot 2^{n-i}$: Pick $i$ stars. The other $n-i$ positions can be freely chosen from $\{0, 1\}$.

By the binomial formula we have:

$$\sum_{i=0}^{n} f_i = \sum_{i=0}^{n} \binom{n}{i} \cdot 2^{n-i} \cdot 1^i = (2 + 1)^n.$$

This is not surprising: Faces correspond to mappings $\{0, 1, \ast\}^n$. 
5.1: Euler–Poincaré (2)

The other equality follows similarly:

\[
\sum_{i=0}^{n} (-1)^i f_i = \sum_{i=0}^{n} \binom{n}{i} \cdot 2^{n-i} \cdot (-1)^i = (2 - 1)^n = 1.
\]

The interesting thing is that this equality holds for all n-dimensional convex polytopes.

In 3-dimensions, this gives Euler’s formula for planar graphs (seen as convex polyhedra):

\[
|V| - |E| + |F| - 1 = 1
\]
5.4: Disjoint Faces as Clauses

Remember: For a clause $C$, the corresponding face is the set of assignments that do not satisfy $C$.

(1) For clauses $C$ and $D$, when are the assignments not satisfying them disjoint?

Claim: If and only if $C$ and $D$ have complementary literals.

If $C$ and $D$ have complementary literals, any assignment not satisfying $C$ satisfies $D$.

If $C$ and $D$ have no complementary literals, then $C \cup D$ is a clause. An assignment not satisfying $C \cup D$ satisfies neither $C$ nor $D$. 
5.4: Disjoint Faces as Clauses (2)

(2) Let $C$ and $D$ be faces of the $V$-cube and let $\phi_C, \phi_D : V \rightarrow \{0, 1, *\}$ be the mappings that induce them.

Suppose there is exactly one position $\hat{i}$ where $\{\phi_C(\hat{i}), \phi_D(\hat{i})\} = \{0, 1\}$. This means that $C$ and $D$ are disjoint and “neighboring”.

In the clauses, such a position corresponds to complementary literals. Remember: Resolution is possible if two clauses have exactly one pair of complementary literals.
Build a new face $E$ with

$$
\phi_E(i) := \begin{cases}
    *, & \{\phi_C(i), \phi_D(i)\} = \{\ast\} \\
    *, & \{\phi_C(i), \phi_D(i)\} = \{0, 1\} \text{ (i.e. } i = \tilde{i}) \\
    0, & \{\phi_C(i), \phi_D(i)\} \in \{\{0\}, \{0, \ast\}\} \\
    1, & \{\phi_C(i), \phi_D(i)\} \in \{\{1\}, \{1, \ast\}\}
\end{cases}
$$

Then $E \subseteq C \cup D$ (why?). If $\{0, 1\}$ never occurred, then $E \subseteq C$ and $E \subseteq D$.

If $\{0, 1\}$ occurred more than once, then $E \not\subseteq C \cup D$.

This corresponds exactly to resolution!
5.5: Not Equivalent to 3 Clauses

Let $V = \{x, y, z\}$. We do a case distinction on the number of non-satisfying assignments.

At most 3 non-satisfying assignments: easy, just use 3-clauses

At least 7 non-satisfying assignments: easy (for 7, use a 1-clause, a 2-clause and a 3-clause)

4 non-satisfying assignments: exactly $x \oplus y \oplus z$ and $\neg(x \oplus y \oplus z)$ have no equivalent CNF formula with 3 clauses. Otherwise we can use one 2-clause and two 3-clauses.
6 non-satisfying assignments: easy. Either a 2-face is non-satisfying, or both satisfying assignments are antipodal. In both cases we find 3 clauses.
For 5 non-satisfying assignments, the argument is a bit more complicated:

If two disjoint edges are non-satisfying, we can use two 2-clauses and one 3-clause.
Otherwise we actually need 4 clauses: 1-clause would gives rise to two disjoint edges, so we have 2-clauses and 3-clauses.

One 3-clause and two 2-clauses gives two disjoint edges or only 4 unsatisfying assignment.

Hence we need three 2-clauses: If no pair of them is disjoint, they cannot cover 5 unsatisfying assignments (why?)

Such a case looks as follows:

![The Cube](image-url)
5.6: Reformulation of Theorem 2.4

A CNF formula $F$ is a collection $\mathcal{F}$ of faces in the cube.

A 2-satisfiable $F$ implies that $\mathcal{F}$ does not contain the whole cube nor any two disjoint facets.

Theorem 2.4 then says that for any collection of faces $\mathcal{F}$ in the cube satisfying the conditions above there is a vertex of the cube contained in at most a fraction of $1-\Phi$ of all faces of $\mathcal{F}$. 
5.7: Edge-connectivity of the $n$-cube

One direction is trivial: if we remove the $n$ edges incident to a vertex $v$, this disconnects $v$ from the remaining cube.

Now suppose we remove $n$ edges not incident to a common vertex. We prove by induction that this leaves the $n$-cube connected.

For $n = 3$, we can inspect all cases to see that the statement holds.
5.7: Edge-connectivity of the $n$-cube (2)

Now suppose the statement holds for some fixed $n - 1 \geq 3$.

In the $n$-cube, consider a partition into facets $A$ and $B$ defined, e.g., by the last coordinate. If all $n$ removed edges run inside $A$, then $B$ stays connected and any vertex from $A$ remains connected to $B$, leaving the whole cube connected. The same holds by symmetry for $B$. 
5.7: Edge-connectivity of the \( n \)-cube (3)

If in one of the facets, say \( A \), \( n - 1 \) edges are removed, this leaves \( A \) (and thus the whole \( n \)-cube) connected unless all \( n - 1 \) edges are incident to a common vertex \( v \in A \) according to the induction hypothesis. In the latter case, \( \{v\} \) and \( A \setminus \{v\} \) are the connected components of \( A \).

But, unless all \( n \) edges are incident to \( v \), \( v \) remains connected to \( A \setminus \{v\} \) via \( B \).

The only case left is that in each facet, at most \( n - 2 \) edges are removed. But then both facets remain connected and there remains at least one edge running between them. \( \Box \)
5.9: Vertex-connectivity of the $n$-cube

Once more proceed by induction. Suppose the removal of at most $n - 1$ vertices could disconnect the $n$-cube.

Consider a partition into two facets $A$ and $B$.

Either all vertices are removed from one side, w.l.o.g. $A$. In this case, $B$ remains connected and all vertices still present in $A$ are connected to $B$ by an edge.
5.9: Vertex-connectivity of the $n$-cube (2)

Otherwise, some vertices are removed from $A$ and some vertices are removed from $B$. Both parts remain connected by the induction hypothesis and since for $n \geq 1$, the number of edges between $A$ and $B$ is larger than $n - 1$, at least one edge remains to connect the two parts.
5.10: Hitting-set for Non-Satisfaction

We are asked to find a code such that every \((n - 2)\)-dimensional face is hit (contains at least one codeword). There are \(\Theta(n^2)\) such faces/ clauses.

If we pick a random codeword, then each fixed face is hit with probability \(\frac{1}{4}\). Therefore the expected number of missed faces when drawing random codewords \(r\) times is

\[
\Theta(n^2) \cdot \left(\frac{3}{4}\right)^r
\]

which can be brought down to a value smaller than one by appropriately choosing \(r = \theta(\log n)\). If the expectation is smaller than one, there exists a realisation of such a code hitting all faces.
5.10: Hitting-set for Non-Satisfaction (2)

How to construct such a hitting-set? We first consider $n = 2^r$.

For $r = 1$, let $A_1 = \{\{0, 0\}, \{0, 1\}, \{1, 0\}, \{1, 1\}\}$.

Suppose we have a hitting set $A_{r-1}$ for $2^{r-1}$ variables.

Let $A_r = \{s \circ s \mid s \in A_{r-1}\} \cup \{0^{(2^{r-1})} \circ 1^{(2^{r-1})}, 1^{(2^{r-1})} \circ 0^{(2^{r-1})}\}$.

To see that $A_r$ is a hitting set, do a case distinction on the two defining positions of the face (\(=\) the variables of the corresponding clause).
We have $|A_r| = |A_{r-1}| + 2$, so $|A_r| = 2r + 2 = 2\log n + 2$.

For arbitrary $n$ use $2^r > n$ and cut off the strings. $(n-2)$-faces in $\{0,1\}^n$ correspond to $(2^r - 2)$-faces in $\{0,1\}^{2^r}$.

Easier to see: 2-clauses over $n$ variables are also 2-clauses over $2^r > n$ variables, but not vice versa.