Exercise 1 (Rigid k-coloring) (50 points)

A proper \( k \)-coloring of a graph is called rigid, if for every vertex of the graph all colors (except for its own) appear in its neighbourhood. In this exercise we look into algorithms that find a rigid \( k \)-coloring. Let \( n \) denote the number of vertices in the graph.

a) Provide a randomized algorithm that produces a rigid \( k \)-coloring for a graph in expected time \( O(k^{(1-1/k)n} \cdot \text{poly}(n)) \), provided it exists.

b) Provide a deterministic algorithm that produces a rigid \( k \)-coloring for a graph in time \( O((k - 1)^n \cdot \text{poly}(n)) \), provided it exists.

c) For any \( \epsilon \in (0, 1/k) \) provide a randomized algorithm that produces a rigid \( k \)-coloring for a graph in expected time \( O((k - 1)^{(1-1/k+\epsilon)n} \cdot \text{poly}(n)) \), provided it exists and assuming the graph is connected.

Hint: Construct a \( k \)-ary covering set of polynomial size. Prove that having the radius \((1 - 1/k + \epsilon)n\) works for this purpose and then use your algorithm from part b) appropriately. See lecture notes p. 155-156 for the definition and properties of \( k \)-ary covering codes. You can use the results from there. You might be able to relax the connectedness assumption somewhat, but having it may make things a little bit easier.

d) There is a way to decide whether a graph is \( k \)-colorable in time \( O(2^n \text{poly}(n)) \) as described by Björklund et al. [2009]. Briefly – and in your own words – describe the main components of their proof in the case of \( k \)-coloring. You don’t need to read the whole paper to do this (but you can: it is really interesting and not too hard!) as the idea for the case of \( k \)-coloring is contained within a few pages. We could also try to use their ideas to decide the existence of a rigid \( k \)-coloring in
comparable time, or at least in time better than just running the algorithms developed in parts a)-c). I don’t know how to do it, but as a bonus question (for some bonus points) show how to decide rigid $k$-coloring relatively fast by using ideas from [Björklund et al. 2009] or from somewhere else.

**Exercise 2 (Properties of the cube) (30 points)**

a) Show that the faces of the cube satisfy a strong Helly-type property: That is, let $\phi_1, \ldots, \phi_m \in \{0, 1, *\}^n$ be faces of the $n$-dimensional Hamming cube. Show that if $\phi_i \cap \phi_j \neq \emptyset$ for all $1 \leq i, j \leq m$ then $\phi := \phi_1 \cap \ldots \cap \phi_m \neq \emptyset$ and furthermore, $\phi$ itself is a face of the Hamming cube.

b) Let $F$ be a CNF formula with $\Box \notin F$. Let the faces of the Hamming cube corresponding (in the sense of Chapter 5 of the lecture notes) to the clauses of $F$ satisfy the properties from part a). Show that $F$ is satisfiable.

c) Let $S \subseteq \{0, 1\}^n$ be nonempty. Show that there exists a tiling of the $n$-dimensional Hamming cube $\{0, 1\}^n$ into disjoint faces $\phi_1, \ldots, \phi_{|S|}$ such that each $\phi_i$ contains exactly one element of $S$.

**References**