Eidgenössische Technische Hochschule Zürich
Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Swiss Federal Institute of Technology Zurich

Institut für Theoretische Informatik
Emo Welzl, Jiří Matoušek

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Theoretische Informatik (Kernfach) *SS 2004*
Exercise Set 5

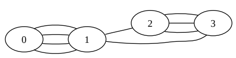
Erinnerung:
Am 30.04. findet die Semesterzwischenprüfung statt. Studenten eingeschrieben in die Übungsgruppen 1–5 legen die Prüfung in NO C 3, Punkt 9:00 bis Punkt 10:00 ab, Studenten eingeschrieben in die Übungsgruppen 6,7 in HG G 3, Punkt 10:00 bis Punkt 11:00.

Exercise 1

An undirected multigraph $G = (V, E)$ is a graph where two (distinct) vertices can be connected by more than one edge, i.e. E is a multiset of two-element sets (no loops). Such a graph can be represented by its adjacency matrix $A_G \in \mathbb{N}^{V \times V}$ where

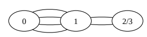
$$A_G(u, v) = \text{number of edges between } u \text{ and } v.$$

Given an edge $\{u, v\}$ in G a contraction of this edge results in a new graph G' with vertex set $V' = V \setminus \{u, v\} \cup \{\boxed{uv}\}$, where $\boxed{uv}$ is a new vertex. The edges in G not connected to u or v survive in G', the edges between u and v vanish and edges from a third vertex w to u or v become edges $\{w, \boxed{uv}\}$.

For example the graph 

has adjacency matrix

$$\begin{pmatrix} 0 & 4 & 0 & 0 \\ 4 & 0 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 1 & 3 & 0 \end{pmatrix}$$

and if one contracts edge $\{2, 3\}$ the resulting graph is 

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Assume for an algorithm on such multigraphs we need three operations:

1. Determine the number of edges between two given vertices.
2. Select an edge of G uniformly at random.
3. Contract a given edge of G.

As the algorithm will use all three operations quite frequently we want to store G in an efficient way.

Describe a data structure together with algorithms implementing these three operations. The goal is to perform the first operation in time $O(1)$ and the other two operations in time $O(n)$. (Here n is the number of vertices of the current G. Note that by contracting an edge the number of vertices decreases.)

Exercise 2

Prove Proposition 1.19 in the lecture notes from Hall's theorem.

Exercise 3

Let (G, s, t, c) be a network with all capacities equal to 1. Assume that there is no s-t-cut with capacity k or smaller ($k > 1$ integer). Prove that there are directed s-t-paths P_1, P_2,..., P_k such that P_i and P_j have no common edges for $1 \leq i < j \leq k$. (Two paths may have common vertices.)

Hint: Consider a suitable maximum flow and extract the paths by induction.

Exercise 4

For $i, n \in \mathbb{N}, 1 \leq i \leq n$, let $R_n^{(i)}$ be the depth of the key of rank i in a *rightist* random search tree for n keys. (Recall the definition from Exercise 1.19, p. 12 in the hand-out *Random(ized) Search Trees.*)

Determine $E[2^{R_n^{(n)}}]$.

Exercise 5

For a random search tree on n nodes and $i, j, k \in \{1, \ldots n\}, i \leq j$, define the random indicator variable

$$C_{i,j}^k = [\text{node } k \text{ is common ancestor of nodes } i \text{ and } j]$$

(where "node k" stands for "node holding key of rank k", etc.).

Determine $\Pr[C_{i,j}^k = 1]$ for $i, j, k \in \{1, \ldots n\}$, $i \leq j$.

Hint: You'll have to discriminate the cases $k < i \leq j$, $i \leq k \leq j$, and $i \leq j < k$.

Due-date:29.04.04 in the lecture