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Theoretische Informatik (Kernfach) SS 2004
Exercise Set 7

Exercise 1

A graph G has vertex set $\{1, 2, \ldots, 4n\}$, and vertices i and j are connected by an edge if

(i) $1 \le i, j \le 2n$, or

(ii) $2n + 1 \le i, j \le 4n$, or

(iii) $i \le n$ and $j = i + 2n$.

What is the size of a minimum cut in G? Prove your answer!

If you fail for general n try to solve the exercise for $n = 3$.

Exercise 2

For a graph G a vertex cut is a set C of vertices, such that removing the vertices in C and the incident edges makes the graph disconnected. Note that a complete graph has no vertex cut. Let $\kappa_e(G)$ be the size of a minimum edge cut in G and $\kappa_v(G)$ the size of a minimum vertex cut. (Per definition for the complete graph on n vertices set $\kappa_v(K_n) := n - 1$.)

(a) Prove that $\kappa_e(G) \ge \kappa_v(G)$.

(b) Does there exist a function $f : \mathbf{N} \to \mathbf{N}$, such that for any graph G the inequality $\kappa_e(G) \le f(\kappa_v(G))$ holds?

Exercise 3 (One-Level Bootstrapping)

Let algorithm Contract be

```
CONTRACT(G, t):
  for i ← n downto t + 1
      for random e ∈ E(G)
      G ← G/e
  return G
```

Note that Contract(G, t) executes $n - t - 1$ contractions and thus ends up with a graph with t vertices. (In contrast to the version in the script where by mistake the for-loop is executed one time too much.)

We consider the following algorithm for minimum cut, with an input graph G on n vertices and with additional parameters r and t:

```
ONELEVELBOOT(G, t, r):
  H ← CONTRACT(G, t)
  mink ← ∞
  for i ← 1 to r
      X ← CONTRACT(H, 2)
      if size(X) < mink
          mink ← size(X)
          minX ← X
  return minX
```

In the following, you are asked to analyze the behavior of this algorithm for various settings of r and t. The goal is to make rough calculations to realize what is going on. You need not worry about integer parts, say. For the probability of success of Contract(G, t) use the bound derived in class, which is roughly t^2/n^2, $t \geq 2$.

(a) Consider OneLevelBoot called with $t = n/2$ and $r = 2$. How do the probability of success and the running time change compared to the basic guessing algorithm Contract$(G, 2)$?

(b) Consider OneLevelBoot called with $t = \sqrt{n}$ and $r = n$. Estimate the probability of success. How many times do we need to repeat this algorithm, in order to make the probability of success at least $\frac{1}{2}$? What is the total running time of these repetitions?

(c) Consider now $t = n^\alpha$, $r = n^\beta$, where $\alpha \in (0, 1)$ and $\beta > 0$ are constants. Again estimate the success probability and number N of repetitions needed to make the success probability at least $\frac{1}{2}$. What choice of α and β give the best total running time? (If you cannot determine the very best ones, at least give the best ones you can find.)

Exercise 4 (Challenge)

Prove that no graph on n vertices has more than $\binom{n}{2}$ minimum cuts.
Hint: What happens if Karger's algorithm avoids contracting edges from a given minimum cut?

Due-date: 13.05.04 in the lecture