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Exercise Set 8

Exercise 1

Given a finite set S of rational numbers and positive integers d and n, $d \geq |S|$, find a polynomial $p(x_1, x_2, \ldots, x_n)$ of degree d for which the Schwartz–Zippel theorem is tight. That is, the number of n-tuples $(r_1, \ldots, r_n) \in S^n$ with $p(r_1, \ldots, r_n) = 0$ is $d|S|^{n-1}$.

Exercise 2

Consider a modified checker for matrix multiplication (over the reals): Instead of choosing a random vector with components 0 and 1, choose a random vector with components drawn from $\{0, 1, \ldots, N-1\}$ uniformly and independently at random. Show that the probability of failure (declaring an incorrect multiplication correct) is at most $\frac{1}{N}$.

Exercise 3

Suppose that we have an algorithm for testing the existence of a perfect matching in a given graph, with running time at most $T(n)$ for any n-vertex graph.

(a) Explain how repeated calls to the algorithm can be used to find a perfect matching if one exists. Estimate the running time of the resulting algorithm.

(b) How can the algorithm be used for finding a maximum matching in a given graph?

Exercise 4

Suppose that a polynomial $p(x_1, \ldots, x_n, y)$ in $n+1$ variables is given by a black box that, given concrete values for $x_1, \ldots, x_n$ and y, returns the value of the polynomial. Let m be the maximum degree of y in p and write $p(x_1, \ldots, x_n, y) = \sum_{i=0}^{m} y^i p_i(x_1, \ldots, x_n)$. Given an integer k and numbers $r_1, \ldots, r_n$, how can we compute $p_k(r_1, \ldots, r_n)$ (using only the black box)? How can we test whether $p_k(x_1, \ldots, x_n)$ is a nonzero polynomial? For simplicity, assume that everything happens over the rationals.

Exercise 5

Consider a bipartite graph in which some edges are colored red and some blue. Extend the randomized algorithm discussed in class to handle the following problem: Given such a colored bipartite graph and an integer k, is there a perfect matching that contains exactly k red edges?

Hint: Use Exercise 4.

(*Remark:* No polynomial-time deterministic algorithm for this problem seems to be known.)

Exercise 6

Let $n \in \mathbf{N}$. The sign $\mathrm{sign}(\pi)$ of a permutation π of $\{1..n\}$ can be defined, e.g., by

$$\mathrm{sign}(\pi) = \prod_{1 \leq i < j \leq n} \frac{\pi(i) - \pi(j)}{i - j}.$$

Recall that for two permutations π and σ of $\{1..n\}$ we have

$$\mathrm{sign}(\pi \circ \sigma) = \mathrm{sign}(\pi) \cdot \mathrm{sign}(\sigma),$$

where $\circ$ is the concatenation of π and σ, i.e $\pi \circ \sigma(x) = \pi(\sigma(x))$.

Furthermore recall that for a permutation π with an odd cycle the permutation $r(\pi)$ has exactly the smallest cycle reversed.

(a) Show that for a permutation π which consists of only one odd cycle and no even cycle the signs of π and $r(\pi)$ are equal.

(b) Show that for a permutation π with an odd cycle the signs of π and $r(\pi)$ agree.

Exercise 7

Let A be a $n \times n$ matrix with 0/1-entries. For $1 \le i, j \le n$ let $\epsilon_{i,j}$ be independent random variables, $\epsilon_{i,j} \in_{u.a.r} \{-1, +1\}$. Let B be the random matrix with $b_{i,j} = \epsilon_{i,j} \cdot a_{i,j}$. In other words, to get B from A we randomly assign signs to the entries of A.

(a) Show that $\mathbf{E}[\det B] = 0$.

(b) Show that $\mathbf{E}[(\det B)^2] = \mathrm{per}(A)$. (Challenge)

Due-date:27.05.04 in the lecture