

## Topological Methods in Combinatorics and Geometry FS 08

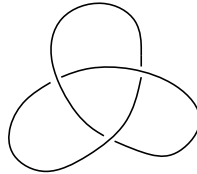
### (Optional) Easter Problem Set

Course webpage: <http://www.ti.inf.ethz.ch/ew/courses/Top08/>

Due date: April 3, 2008

This problem set is optional. If you don't have enough points from previous problem sets, you can solve the problems and thus get more points.

**Exercise 1.** Decide, which pairs of the following spaces are homeomorphic (include, not necessarily formal, proofs):  $(0, 1)$ ,  $[0, 1]$ ,  $\mathbb{S}^1$ ,  $\mathbb{R}$ , and the trefoil knot  $K$  drawn on the figure.



**Exercise 2.** Let  $X$  and  $Y$  be topological spaces,  $f: X \rightarrow Y$  a continuous mapping, and  $M, N \subseteq X$ . Prove or disprove the following statements:

- (a) If  $M$  is closed then  $f(M)$  is closed.
- (b) If  $M$  is open then  $f(M)$  is open.
- (c) If  $M$  is connected then  $f(M)$  is connected.
- (d) If  $M$  is unconnected then  $f(M)$  is unconnected.
- (e) If  $M$  is closed and  $N$  compact then  $M \cap N$  is compact.

**Exercise 3.** Let  $T$  be a torus. Show that there is no continuous injection  $f: T \rightarrow \mathbb{S}^2$ . *Hint:* Show that the graphs  $K_5$  and  $K_{3,3}$  embed into  $T$  and use the Kuratowski theorem.

**Exercise 4.** (a) Find a triangulation of the cube  $[0, 1]^3$  using just five 3-simplices. (It is sufficient just to draw a figure.) *Hint:* Try to find a regular tetrahedron spanned by some of the vertices of the cube.

- (b) Prove that every triangulation of the cube  $[0, 1]^3$  uses at least five 3-simplices. *Hint:* Derive that each simplex covers at most  $\frac{1}{4}$  of the surface of the cube and in such case it is unique (up to isometries of the cube).

**Exercise 5.** For positive integer  $n$  prove or disprove the following statement: Whenever  $A_1, A_2, \dots, A_{n+1}$  is a cover of  $S^n$  by  $n+1$  sets, each of which can be obtained from open sets by finitely many set-theoretic operations (union, intersection, difference), then there is  $i$  with  $(A_i) \cap (-A_i) \neq \emptyset$ .