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Topological Methods in Combinatorics and Geometry FS 08

Problem Set 1

Course webpage: `http://www.ti.inf.ethz.ch/ew/courses/Top08/`

Due date: February 28, 2008

Exercise 1 (**Kneser graphs**)**.** Recall the definition of the Kneser graph $\mathrm{KG}(\binom{[n]}{r})$, for integers $1 \le r \le n/2$: the vertices are the r-element subsets of $[n] = \{1, \ldots, n\}$, and $\{A, B\}$ is an edge iff $A \cap B = \emptyset$. Show that the chromatic number of the Kneser graph satisfies $\chi(\mathrm{KG}(\binom{[n]}{r})) \le n - 2r + 2$.

Exercise 2 (**Glueing lemma for maps**)**.** Let X and Y be topological spaces.

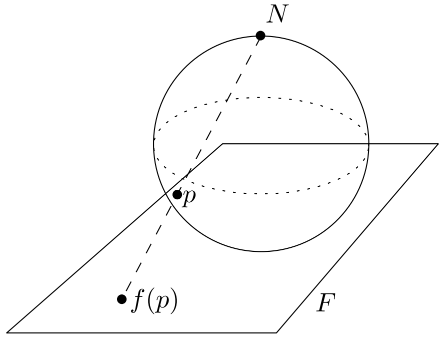
(a) Show that a map $f : X \to Y$ is continuous iff the preimage $f^{-1}[B]$ of every closed set $B \subseteq Y$ is closed in X.

(b) Let $A_1, \ldots, A_n$ be closed sets in X such that $X = A_1 \cup \ldots \cup A_n$. Suppose that we have continuous maps $f_i : A_i \to Y$ (with respect to the subspace topology on the A_i) that agree on overlaps, i.e, $f_i|_{A_i \cap A_j} = f_j|_{A_i \cap A_j}$, $1 \le i, j \le n$. Show that one can "glue them together", i.e., that there is a continuous map $f : X \to Y$ with $f|_{A_i} = f_i$ for all i.

Exercise 3 (**A homeomorphism**)**.** Consider the 2-sphere

$$\mathbb{S}^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$$

and the "north pole" $N = (0, 0, 1)$. Show that $\mathbb{S}^2 \setminus \{N\} \cong \mathbb{R}^2$. *Hint.* Consider the 2-dimensional plane $F = \{x \in \mathbb{R}^3 : x_3 = -1\} \cong \mathbb{R}^2$ (why is it homeomorphic?) and the stereographic projection f from N to F, as in the following figure. Generalize this to higher dimensions.

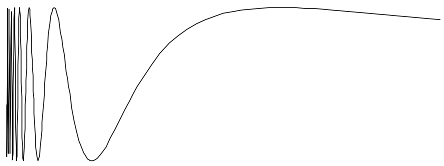


Exercises 4 and 5 (on the following page) may be more challenging. Think of them as a motivation to read the relevant sections in Jänich's book.

Exercise 4 (**Another homeomorphism**)**.** Let $K \subseteq \mathbb{R}^d$ be a compact convex set with nonempty interior. Show that the boundary ∂K is homeomorphic to $\mathbb{S}^{d-1}$. (You may use the facts about compactness in Section 1.8 of Jänich's book.)

Exercise 5 (Connectivity). A topological space X is called *connected* if it cannot be written as the disjoint union of two nonempty disjoint open subsets, i.e., if $X = U \cup V$ with both U and V open in X and nonempty implies $U \cap V \neq \emptyset$. The space X is called *path connected* if for every pair $x, y \in X$, there is a *path* from x to y, i.e., a continuous map $f : [0,1] \to X$ with $f(0) = x$ and $f(1) = y$ (see Section 1.7 in Jänich's book). Consider the following subspace of $\mathbb{R}^2$:

$$X = \{(x, \sin(1/x) : x > 0\} \cup \{(0, y) : -1 \leq y \leq 1\}.$$



Show that X is connected but not path connected.