ETH Eidgenössische Technische Hochschule Zürich
Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Swiss Federal Institute of Technology Zurich

Institute of Theoretical Computer Science
Jiří Matoušek, Martin Tancer, Uli Wagner

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Topological Methods in Combinatorics and Geometry FS 08

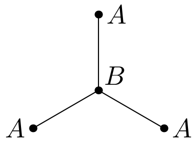
Problem Set 11

Course webpage: `http://www.ti.inf.ethz.ch/ew/courses/Top08/`

Due date: May 29, 2008

Exercise 1. Find a simplicial complex K such that $\|\mathsf{K}_{\Delta}^{*2}\|$ is homeomorphic to $S^1 \times [0,1]$.

Exercise 2 (n-dimensional space that does not embed into $\mathbb{R}^{2n}$)**.** Let A be a three point discrete space and B be a singleton space. The 3-*star* is the space $Y = A * B$ drawn in the picture (we consider A and B as subspaces of Y).



Next, we define the space[1]

$$N = \left\{(y_1, y_2, \dots . y_{n+1}) \in Y^{n+1} \mid y_i \in A \text{ for at least one } i \in [n+1]\right\}.$$

The topology on N is the inherited topology of N as a subspace of the Cartesian product[2] Y^{n+1}.

(a) Find a map $f\colon B^2 \to Y$ such that $f(\mathbf{x}) \neq f(-\mathbf{x})$ for every $\mathbf{x} \in S^1 = \partial B^2$.

(b) Prove that Y^{n+1} does not embed into $\mathbb{R}^{2n+1}$.

(c) Prove that if N embeds into $\mathbb{R}^k$ then Y embeds into $\mathbb{R}^{k+1}$, and thus conclude that N does not embed into $\mathbb{R}^{2n}$.

Exercise 3. Let $V_{n,2} = \{(\mathbf{v}_1, \mathbf{v}_2) \in (S^{n-1})^2 \mid \langle \mathbf{v}_1, \mathbf{v}_2 \rangle\} \subset \mathbb{R}^{2n}$ be the Stiefel manifold of pairs of unit orthogonal vectors, $n \geq 1$. Let ν be the $\mathbb{Z}_2$-action given by $(\mathbf{v}_1, \mathbf{v}_2) \to (-\mathbf{v}_1, -\mathbf{v}_2)$.

(a) Show that $\mathrm{ind}_{\mathbb{Z}_2}(V_{2,n}) \leq n-1$.

(b) Let n be even. Exhibit a $\mathbb{Z}_2$-map $S^{n-1} \to V_{n,2}$, thereby proving that $\mathrm{ind}_{\mathbb{Z}_2}(V_{n,2}) = n-1$.

[1] The space N can be seen as a geometric realization of some n-dimensional simplicial complex, but you are not supposed to prove it.

[2] If we consider Y as subspace of $\mathbb{R}^2$, then we can see Y^{n+1} as a subspace of $\mathbb{R}^{2(n+1)}$.