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Topological Methods in Combinatorics and Geometry FS 08

Problem Set 9

Course webpage: `http://www.ti.inf.ethz.ch/ew/courses/Top08/`

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Exercise 1. Consider the boundary of the $(n-1)$-dimensional simplex as an abstract simplicial complex K; i.e., $\mathsf{K} = 2^{[n]} \setminus \{[n]\}$. Let $\mathsf{L} = \mathrm{sd}(\mathsf{K})$ be the first barycentric subdivision of K. (Thus, the vertex set of L consists of all proper nonempty subsets of $[n]$.) We define a simplicial map $\nu\colon V(\mathsf{L}) \to V(\mathsf{L})$ by setting, for every vertex $F \in V(\mathsf{L})$, $\nu(F) = [n] \setminus F$. Prove that $(\|\mathsf{L}\|, \|\nu\|)$ is a free $\mathbb{Z}_2$-space.

Exercise 2. Let K be a simplicial complex, and let ν be a free simplicial $\mathbb{Z}_2$-action on K. Prove that $F \cap \nu(F) = \emptyset$ for every $F \in \mathsf{K}$.

Exercise 3. Let $\mathsf{B}(G)$ denote the box complex of a graph G. Determine topologically $\mathsf{B}(K_4)$ and $\mathsf{B}(C_5)$ (C_5 denotes the cycle on 5 vertices); i.e., find a well-known topological spaces that are homeomorphic to these complexes.

Exercise 4. (a) Let $p(x_1, x_2, \ldots, x_n) = p(\mathbf{x})$ be a nonzero polynomial in n variables. Show that the zero set $Z(p) = \{\mathbf{x} \in \mathbb{R}^n | p(\mathbf{x}) = 0\}$ is nowhere dense, meaning that any open ball B contains an open ball B' with $B' \cap Z(p) = \emptyset$.

(b) Check that a finite union of nowhere dense sets is nowhere dense.

(c) Let $\sigma = \mathrm{conv}\{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_{n+1}\}$ be an $(n+1)$-dimensional simplex. Let $h\colon \sigma \to \mathbb{R}^n$ be an affine map (i.e., a map of the form $\mathbf{x} \to A\mathbf{x}^T + \mathbf{b}$, where A is an $n \times (n+1)$ matrix and $\mathbf{b} \in \mathbb{R}^n$). If each h is represented by $(h(\mathbf{0}), h(\mathbf{e}_1), \ldots, h(\mathbf{e}_{n+1})) \in \mathbb{R}^{(n+2)n}$, show that the set of maps such that $h^{-1}(\mathbf{0})$ intersects a face of σ of dimension smaller than n form a nowhere dense set. Hint: For each face of σ of small dimension, write down a determinant that becomes 0 for all maps such that $h^{-1}(\mathbf{0})$ intersects that face.