

Some Basic Facts About Measures, and the Ham-Sandwich Theorem

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We recall some basic definitions and facts from measure theory.

Definition 1. Let X be a set. A σ -algebra on X is a family $\mathcal{A} \subseteq 2^X$ of subsets of X , which are called *measurable subsets*, such that \mathcal{A} nonempty and closed under complements and countable unions and intersections. More formally, one assume that \mathcal{A} satisfies the following axioms:

1. $X \in \mathcal{A}$.
2. If $A \in \mathcal{A}$ then $X \setminus A \in \mathcal{A}$.
3. If $\{A_i\}_{i=1}^{\infty} \subseteq \mathcal{A}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

The pair (X, \mathcal{A}) is sometimes called a *measurable space*.

Note that Axioms 2 and 3 and De Morgan's rules imply that a σ -algebra is also closed under countable intersections. Another immediate consequence of the axioms is that $A \setminus B = A \cap (X \setminus B)$ is measurable whenever both A and B are.

It is easy to check that arbitrary intersections of σ -algebras are again σ -algebras, i.e., if $\{\mathcal{A}_i : i \in I\}$ is an arbitrary family of σ -algebras on X (where I is an arbitrary nonempty index set) then $\bigcap_{i \in I} \mathcal{A}_i$ is again a σ -algebra. Thus, if \mathcal{S} is an arbitrary family of subsets of X , then one can define the σ -algebra generated by \mathcal{S} as the smallest σ -algebra containing \mathcal{S} , i.e., as

$$\bigcap_{\substack{\mathcal{S} \subseteq \mathcal{A} \subseteq 2^X \\ \mathcal{A} \text{ } \sigma\text{-algebra}}} \mathcal{A}.$$

An important example is the following: if X is a topological space then the *Borel σ -algebra* on X is the σ -algebra generated by the collection of open sets.

Definition 2. Let (X, \mathcal{A}) be a measurable space. A *measure* is a function $\mu : \mathcal{A} \rightarrow [0, \infty] = \mathbb{R}_{\geq 0} \cup \{+\infty\}$ that assigns to every measurable subset a positive number or ∞ such that the following axioms are satisfied:

1. $\mu(\emptyset) = 0$.
2. If $\{A_i\}$ is a finite or countably infinite family of *pairwise disjoint* measurable sets (i.e., $A_i \cap A_j = \emptyset$ whenever $i \neq j$) then $\mu(\bigcup_i A_i) = \sum_i \mu(A_i)$. This property is sometimes called *countable additivity*.

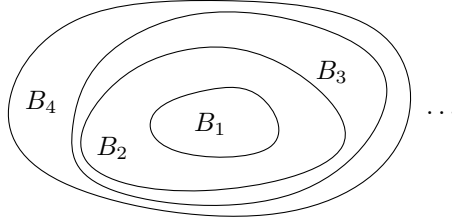
The triple (X, \mathcal{A}, μ) is called a *measure space*. The measure is called *finite* if $\mu(X) < \infty$.

We list some basic properties of measures. These properties are easy to prove, using just the basic definitions.

Lemma 3. Let (X, \mathcal{A}, μ) be a measure space.

1. If $A \subseteq B$ are measurable then $\mu(A) \leq \mu(B)$.
2. If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ is an increasing sequence of measurable sets then $\mu(\bigcup_i A_i) = \lim_i \mu(A_i) = \sup_i \mu(A_i)$.
3. If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ is a decreasing sequence of measurable sets and if $\mu(A_i) < \infty$ for some i then $\mu(\bigcap_i A_i) = \lim_i \mu(A_i) = \inf_i \mu(A_i)$.

Proof. Let $A \subseteq B$ be measurable. Then $\mu(B) = \mu(A) + \mu(B \setminus A)$, and $\mu(B \setminus A) \geq 0$. This implies the first assertion. For the second part, note that if $A_1 \subseteq A_2 \subseteq \dots$ is an increasing sequence of measurable subsets, then by the first part, their measures $\mu(A_i)$ form an increasing sequence in $[0, \infty]$, which converges to some limit (possibly ∞), namely $\lim_i \mu(A_i) = \sup_i \mu(A_i)$. Set $A_0 := \emptyset$ and $B_i := A_i \setminus A_{i-1}$, $i \geq 1$. Note that the sets B_i are pairwise disjoint and that $A_i = B_1 \cup \dots \cup B_i$ for all $i \geq 1$. Consequently, also $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} B_i$.



Thus, by countable additivity, $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(B_i) = \lim_{i \rightarrow \infty} \sum_{j=1}^i \mu(B_j) = \lim_i \mu(A_i)$, as desired.

For the third part, note that $\mu(A_i)$ is an decreasing sequence, so the limit exists. Moreover, w.l.o.g., $\mu(A_1) < \infty$ (since we may change or discard any finite initial part of the sequence of sets without affecting either the intersection or the limit). Set $C_i := A_1 \setminus A_i$. Note that $\mu(A_i) + \mu(C_i) = \mu(A_1)$, and since $\mu(C_i) \leq \mu(A_1) < \infty$, we may subtract $\mu(C_i)$ and conclude $\mu(A_i) = \mu(A_1) - \mu(C_i)$. Moreover, the sets C_i form an increasing sequence of sets, so by the second part, $\lim_i \mu(C_i) = \mu(\bigcup_i C_i)$. Furthermore, $\bigcup_i C_i = \bigcup_i (A_1 \setminus A_i) = A_1 \setminus (\bigcap_i A_i)$, so (again using that $\mu(A_1) < \infty$), $\mu(\bigcap_i A_i) = \mu(A_1) - \mu(\bigcup_i C_i) = \mu(A_1) - \lim_i \mu(C_i) = \lim_i \mu(A_1 - C_i) = \lim_i \mu(A_i)$, by Part 2. \square

We also note the following properties which we used in the proof of the Ham-Sandwich theorem.

Lemma 4. 1. If A and B are measurable sets and at least one of them has finite measure then $|\mu(A) - \mu(B)| \leq \mu(A \oplus B)$, where $A \oplus B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference. (Note that if both measures are infinite then the left hand side of the equation is not well defined, since there is no meaningful way to define $\infty - \infty$.)

2. If $\{A_i\}_{i=1}^{\infty}$ is a countable family of measurable sets and if $\mu(\bigcup_i A_i) < \infty$ then

$$\limsup_{i \rightarrow \infty} \mu(A_i) \leq \mu\left(\bigcap_{n=1}^{\infty} \bigcup_{i \geq n} A_i\right)$$

(Recall that for any sequence $\{a_i\}_{i=1}^{\infty}$ of real numbers, $\limsup_{i \rightarrow \infty} a_i = \lim_{n \rightarrow \infty} \sup_{i \geq n} a_i$.)

We remark that one also often uses the notation $\limsup_i A_i := \bigcap_{n=1}^{\infty} \bigcup_{i \geq n} A_i$. Note that an element x belongs to this set iff it is an element of infinitely many sets A_i .

Proof. For the first part, assume w.l.o.g. that $\mu(A) \leq \mu(B)$, so in particular $\mu(A) < \infty$. We have $B \subseteq A \cup (B \setminus A)$, hence $\mu(B) \leq \mu(B \setminus A) + \mu(A \cap B) \leq \mu(B \setminus A) + \mu(A)$. Since $\mu(A \cap B) \leq \mu(A) < \infty$, it follows that $\mu(B \setminus A) = \mu(B) - \mu(A \cap B) \geq \mu(B) - \mu(A)$. Consequently, if $\mu(B) = \infty$ then both $|\mu(A) - \mu(B)| = \infty$ and $\mu(A \oplus B) \geq \mu(B \setminus A) = \mu(B) - \mu(A \cap B) = \infty$, so the statement is true. Thus, we may assume that also $\mu(B) < \infty$, and by a completely symmetric argument as above, we get $\mu(A \setminus B) \geq \mu(A) - \mu(B)$, hence $|\mu(A) - \mu(B)| \leq \max\{\mu(B \setminus A), \mu(A \setminus B)\} \leq \mu(A \oplus B)$.

For the second part, we apply Part 3 of the preceding lemma to the decreasing sequence of sets $D_n = \bigcup_{i \geq n} A_i$. It follows that $\limsup_i \mu(A_i) = \lim_n \sup_{i \geq n} \mu(A_i) \leq \lim_n \mu(D_n) = \mu(\bigcap_n D_n) = \mu(\bigcap_n \bigcup_{i \geq n} A_i)$. \square

Similarly, one can show that

$$\begin{aligned} \liminf_i \mu(A_i) &\geq \mu\left(\underbrace{\bigcup_n \bigcap_{i \geq n} A_i}_{=: \liminf_i A_i}\right) \end{aligned}$$

holds for any countable family of measurable sets (no finiteness assumption on the measure is needed). This is a special case of *Fatou's Lemma* from integration theory. In a sense, this is more basic than the estimate for the limit superior above, but we need the latter for the proof of the Ham-Sanwich Theorem.

In the proof, we will consider functions $\mathbb{S}^d \rightarrow \mathbb{R}$ of the following form. For $u = (u_0, u_1, \dots, u_d) \in \mathbb{S}^d$, define $h(u) = \{x \in \mathbb{R}^d : u_1 x_1 + \dots + u_d x_d = u_0\}$ and $h^-(u) := \{x \in \mathbb{R}^d : u_1 x_1 + \dots + u_d x_d < u_0\}$. Note that $h(u)$ is a hyperplane (and $h^-(u)$ an open halfspace) unless u is the north pole $(1, 0, \dots, 0)$ or the south pole $(-1, 0, \dots, 0)$. For these two special u 's, we have $h(u) = \emptyset$. Moreover, $h^-(u) = \mathbb{R}^d$ and $h^+(u) = \emptyset$ for $u = (1, 0, \dots, 0)$, and vice versa for $u = (-1, 0, \dots, 0)$. Also note that for any u , $h^-(-u) = h^+(u) = \{x : u_1 x_1 + \dots + u_d x_d > u_0\}$.

Lemma 5. *Let μ be a finite Borel measure on \mathbb{R}^d such that $\mu(h) = 0$ for all hyperplanes. Then the function $f : \mathbb{S}^d \rightarrow \mathbb{R}$, $f(u) = \mu(h^-(u))$ is continuous.*

Proof. We use the fact that in a metric space, a function is continuous iff it preserves limits of convergent sequences. More precisely, let $(u^{(n)})_{n=1}^\infty$ be a sequence in \mathbb{S}^d that converges to some point u . We need to show that $f(u^{(n)})$ converges to $f(u)$. (We use upper indices to distinguish the sequence index from coordinate indices. In class, we did this the other way around, i.e., we used superscripts for coordinate indices and subscripts for the index of the sequence.) We are done if we can prove that $\limsup_n |f(u) - f(u^{(n)})| \leq 0$. By Part 1 of Lemma 4, we have

$$|f(u) - f(u^{(n)})| \leq \mu(\underbrace{h^-(u) \oplus h^-(u^{(n)})}_{=: A_n}).$$

By Part 2 of the same lemma, it suffices to show that

$$\mu\left(\bigcap_{N=1}^\infty \bigcup_{n \geq N} A_n\right) = 0.$$

We claim that this last set is contained in $h(u)$. Since $h(u)$ is either empty or a hyperplane, we are done if we can prove this claim. We show the contraposition: let $x \notin h(u)$. First consider $x \in h^-(u)$, i.e., $u_1 x_1 + \dots + u_d x_d < u_0$. Since x is fixed and $u_i = \lim_n u_i^{(n)}$ for $0 \leq i \leq d$, it

follows that for all sufficiently large n , we also have $x \in h^-(u^{(n)})$. Thus, for all sufficiently large n we have $x \in h^-(u) \cap h^-(u^{(n)})$, hence $x \notin A_n = h^-(u) \oplus h^i(u^{(n)})$. Therefore $x \notin \limsup_n A_n$. Similarly, if $x \in h^+(u)$ then $x \in h^+(u^{(n)})$ (whence also $x \notin A_n$) for all sufficiently large n , so again $x \notin \limsup_n A_n$. this finishes the proof of the claim and hence of the lemma. \square

Theorem 6 (Continuous Ham-Sandwich Theorem). *Let μ_1, \dots, μ_d be finite Borel measures on \mathbb{R}^d such that $\mu_i(h) = 0$ for every hyperplane h and all i . Then there exists a hyperplane h that simultaneously bisects all μ_i , i.e.,*

$$\mu_i(h^-) = \mu_i(h^+) = \frac{1}{2}\mu_i(\mathbb{R}^d), \quad 1 \leq i \leq d,$$

where h^+ and h^- are the two open halfspaces defined by h .

Proof. Note that we may assume that at least one $\mu_i(\mathbb{R}^d) > 0$, else any hyperplane trivially bisects all μ_i .

Consider the function $f = (f_1, \dots, f_d) : \mathbb{S}^d \rightarrow \mathbb{R}^d$ with component functions $f_i(u) := \mu_i(h^-(u))$ as defined above. This function is continuous. Therefore, by the Borsuk-Ulam theorem, there exists a point $u^* \in \mathbb{S}^d$ with $f(u^*) = f(-u^*)$. As remarked above, $h^-(-u) = h^+(u)$ for all u , so $f(u^*) = f(-u^*)$ implies $\mu_i(h^-(u^*)) = \mu_i(h^+(u^*))$ for all i . Moreover, if u^* were the north pole or the south pole then $h^-(u^*) = \mathbb{R}^d$ and $h^+(u^*) = \emptyset$, or vice versa. If at least one $\mu(\mathbb{R}^d) > 0$, as we assume, this is impossible. Thus, u^* is neither of the two special points, hence $h(u^*)$ is an honest hyperplane, which simultaneously bisects all μ_i . \square