Exercise 1. Let \((X, d)\) be a compact metric space, and let \(\mathcal{U} = \{U_i | i \in I\}\) be an open cover of \(X\). Show that there is \(\delta > 0\) such that for any \(A \subseteq X\) with \(\text{diam}(A) < \delta\) there is \(i \in I\) with \(A \subseteq U_i\).

Remark: For a subset \(A \subseteq X\), the diameter of \(A\) is \(\text{diam}(A) = \sup\{d(a, a') | a, a' \in A\}\).

The following exercises consider the torus \(T = S^1 \times S^1 = \{(\cos \alpha, \sin \alpha, \cos \beta, \sin \beta) : \alpha, \beta \in [0, 2\pi]\}\).

It might be useful to recall the following alternative description of the torus: It can be obtained from the square \([0, 1]^2\) by identifying opposite sides as indicated below.

Exercise 2 (Torus I). Choose a point \(p \in T\) of the torus \(T\) and construct an explicit deformation retraction of \(T \setminus \{p\}\) onto a graph consisting of two circles intersecting in a point, namely, longitude and meridian circles of the torus.

Exercise 3 (Torus II). Exhibit a triangulation of the torus \(T\). Use as few simplices as you can.

Exercise 4 (Affine Independence). Let \(v_0, v_1, \ldots, v_k\) be points in \(\mathbb{R}^d\). In the lecture we defined \(v_0, v_1, \ldots, v_k\) to be affinely independent if \((v_0, 1), (v_1, 1), \ldots, (v_k, 1)\) are linearly independent. Show that the following conditions are equivalent to this definition:

- The \(k\) vectors \(v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0\) are linearly independent.
- If \(\sum_{i=0}^{k} \alpha_i v_i = 0\) and \(\sum_{i=0}^{k} \alpha_i = 0\) for \(\alpha_0, \alpha_1, \ldots, \alpha_k \in \mathbb{R}\), then \(\alpha_0 = \alpha_1 = \ldots = \alpha_k = 0\).
- The affine hull of \(v_0, v_1, \ldots, v_k\), defined as \(\text{aff}(v_0, v_1, \ldots, v_k) = \{\sum_{i=0}^{k} \alpha_i v_i | \sum_{i=0}^{k} \alpha_i = 1\}\), has dimension \(k\).
Exercise 5 (Triangulations of Prisms).

Let $\sigma$ be a $n$-simplex with vertices $v_0, v_1, \ldots, v_n$ and let $P = \sigma \times [0,1]$ be the $(n+1)$-dimensional “prism” over $\sigma$. For $0 \leq i \leq n$, let $v'_i = (v_i, 0)$ and $v''_i = (v_i, 1)$ the “bottom” and the “top vertex” corresponding to $v_i$. Also let $\sigma_i := \text{conv}\{v'_0, v'_1, \ldots, v'_i, v''_i, \ldots, v''_n\}$.

The sets $\sigma_0, \ldots, \sigma_d$ are $(n+1)$-dimensional simplices. The collection of simplices consisting of the $\sigma_i$’s and all their faces is a simplicial complex, and it is a triangulation of $P$. (The proof of this is not hard, but rather on the lengthy side. Can you think of what needs to be shown? If you feel like it, go on and show it.)

(a) Draw a picture of $P$ for $n = 1, 2$.

(b) Let $\Delta$ be a simplicial complex. Explain how to use the above construction to triangulate $\|\Delta\| \times [0,1]$.

(c) Describe how to use this to obtain a triangulation of the unit cube $C_d = [0,1]^d$. What is the number of $d$-simplices in the triangulation? Draw a picture of $C_d$ for $d = 2, 3$.

(d) In the lecture we also introduced the bottom-vertex triangulation of a convex polytope. Draw a picture of the bottom-vertex triangulation of the cube $C_3$. 

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\includegraphics[width=0.5\textwidth]{prism.png}
\end{center}

\begin{itemize}
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