Topological Methods in Combinatorics and Geometry - FS 11

Problem Set 5

Will be discussed on: April 12, 2011


**Exercise 1.** Recall that a simplicial complex $K$ is a pseudomanifold if the following two conditions hold:

1. Every $(n-1)$-simplex of $K$ is contained in exactly two $n$-simplices.
2. $K$ is connected, i.e. for any two $n$-simplices $F$ and $F'$ there is a chain of adjacent $n$-simplices connecting $F$ and $F'$. Here, being adjacent for two $n$-simplices $F$ and $F'$ means that they share a common $(n-1)$-face.

Show that the barycentric subdivision $sd(K)$ of a pseudomanifold $K$ is also a pseudomanifold. If showing connectedness turns out too laborious, try to do it at least for $\dim(K) = 2$.

**Exercise 2** (2-dimensional Sperner’s Lemma).
Let $T$ be a 2-simplex (triangle) with vertices $v_1, v_2, v_3$. For a subdivision $K$ of $T$, a Sperner labeling of $K$ is a map $\lambda : V(K) \to \{1, 2, 3\}$ that satisfies the following conditions:

1. $\lambda(v_i) = i$ for $i \in \{1, 2, 3\}$.
2. $\lambda(v) \in \{i, j\}$ for a vertex $v \in V(K)$ on the side $v_iv_j$ of the triangle $T$, $i, j \in \{1, 2, 3\}$.

Interior vertices can have arbitrary labels.

Show that for any Sperner labeling $\lambda : V(K) \to \{1, 2, 3\}$ there is a 2-simplex of $K$ with three different vertex labels.

HINT: Consider the following graph: The vertex set is the set of 2-simplices of $K$ plus one additional vertex corresponding to the outside of $T$. Two vertices are adjacent if the corresponding simplices share an edge that is labeled with 1 and 2. The “outside vertex” is connected to all simplices which have an edge on the boundary of $T$ labeled with 1 and 2.

**Exercise 3.** Compute the homology groups of one of the two simplicial complexes pictured below. You can use a computer if you explain how and why you could use it. (The first complex is the union of three triangles and a line segment. In the second picture, vertices with the same labels are supposed to be identified. All drawn triangles belong to the complex. It has 6 vertices, 15 edges and 10 triangles.)
Exercise 4. A graph $G = (V, E)$ can be interpreted as a 1-dimensional simplicial complex. The goal of this exercise is to find graph-theoretical interpretations of the homology groups $H_i(G; \mathbb{Z}_2)$.

(a) Determine $H_0(G; \mathbb{Z}_2)$. HINT: First consider a connected graph $G$.

(b) For $i = 1$: A 1-chain corresponds to a subgraph of $G$. The subgraphs corresponding to 1-cycles can be easily characterized via the degrees of their vertices. Can you find a different description using the graph-theoretical notion of a cycle?

(c) Again for $i = 1$: Let $G$ be a connected graph. Starting with a spanning tree, add the remaining edges one by one, and try to see what happens with $H_1(G, \mathbb{Z}_2)$. 