Topological Methods in Combinatorics and Geometry - FS 11

Graded Homework 3

To be handed in on: May 24, 2011

Course webpage: http://www.ti.inf.ethz.ch/ew/courses/Top11/

Exercise 1 ([4 points]). Prove the following statement using Sperner’s Lemma:
Let $C_1, C_2, \ldots, C_{n+1}$ be closed subsets of $\Delta_n$ satisfying the following property:

$$\text{conv}\{e_i | i \in I\} \subseteq \bigcup_{i \in I} C_i \text{ for all } I \subseteq [n+1].$$

Then there is a point that is contained in all of the $C_i$: $\bigcap_{i \in [n+1]} C_i \neq \emptyset$.

Here, $e_i$ denotes the $i$-th standard vector in $\mathbb{R}^{n+1}$ and we consider the standard $n$-simplex $\Delta_n = \text{conv}\{e_1, \ldots, e_{n+1}\}$. Note that, in particular, the condition implies that the $C_i$ cover $\Delta_n$.

Exercise 2 ([3 points]). For every $n \in \mathbb{N}$, give an example of a free $\mathbb{Z}_2$-space with $\text{ind}_{\mathbb{Z}_2}(X) = n$ that is not $(n-1)$-connected.

Exercise 3 ([5 points]). Let $(X, \nu)$ be a free $\mathbb{Z}_2$-space. Assume, moreover, that $X$ is a metric space.

Show that $\text{ind}_{\mathbb{Z}_2}(X) \geq n$ if and only if for every cover of $X$ by closed sets $F_1, F_2, \ldots, F_{n+1}$ there exists $x \in X$ and $i \in [n]$ such that $x \in F_i$ and $\nu(x) \in F_i$.

Exercise 4 ([6 points]). Let $G = (V, E)$ be a graph. For two vertex sets $A', A'' \subseteq V$ with $A' \cap A'' = \emptyset$, we say that "$G[A', A'']$ is complete" if every vertex in $A'$ is connected to every vertex in $A''$ in $G$, that is, the bipartite induced subgraph of $G$ with vertex classes $A'$ and $A''$ is complete.

We define two $\mathbb{Z}_2$-complexes:

$B'(G)$: The vertices of $B'(G)$ are sets $A' \cup A'' \subseteq V \cup V$ such that $A' \neq \emptyset \neq A''$, $A' \cap A'' = \emptyset$ and $G[A', A'']$ is complete. The simplices of $B'(G)$ are chains of such sets under inclusions. Shorter, we can write

$$B'(G) := \Delta\{A' \cup A'' : A', A'' \subseteq V, A' \neq \emptyset \neq A'', A' \cap A'' = \emptyset, G[A', A''] \text{ is complete}\}$$

where $\Delta$ denotes the order complex. The $\mathbb{Z}_2$-action maps a vertex $A' \cup A''$ to $A'' \cup A'$.

$\tilde{B}(G)$: A vertex of $\tilde{B}(G)$ is an oriented edge $(u, v)$ where $u, v \in V$ and $(u, v) \in E$. The simplices of $\tilde{B}(G)$ are subsets of edge sets of complete bipartite subgraphs $G[A', A'']$ of $G$, where the edges are oriented from $A'$ to $A''$. Shorter,$$
\tilde{B}(G) := \{\tilde{F} \subseteq A' \times A'' : \emptyset \neq A', A'' \subseteq V, A' \cap A'' = \emptyset, G[A', A''] \text{ is complete}\}.$$

Here, $\mathbb{Z}_2$ acts on $\tilde{B}(G)$ by changing the orientations of edges.

Show that $\text{ind}_{\mathbb{Z}_2}(\tilde{B}(G)) = \text{ind}_{\mathbb{Z}_2}(B'(G)) \leq \text{ind}_{\mathbb{Z}_2}(B(G))$ where $B(G)$ denotes the box complex of $G$ that was defined in class.

Exercise 5 ([2 points]). Let $X$ be a $k$-connected space, and let $Y$ be homotopy equivalent to $X$. Show that $Y$ is also $k$-connected.

Also note the bonus problems on the back of this page!
Bonus problems

With the following questions you can gain additional points.

Exercise 6 ([3 points]). Consider a continuous map \( f : X \to Y \) between topological spaces \( X \) and \( Y \). Assume that there are continuous maps \( g, h : Y \to X \) such that \( f \circ g \sim \text{id}_Y \) and \( h \circ f \sim \text{id}_X \). Show that \( f \) is a homotopy equivalence. More generally, show that \( f \) is a homotopy equivalence if there are maps \( g, h : Y \to X \) such that \( f \circ g \) and \( h \circ f \) are homotopy equivalences.

Exercise 7 ([3 points]). Let \((X, \nu)\) be a CI\(_2\)-space.

(a) Show that \( \text{coind}_{\text{CI}_2}(X) \leq \text{ind}_{\text{CI}_2}(X) \).

(b) Let also \( Y \) be a CI\(_2\)-space, and assume that \( \text{coind}_{\text{CI}_2}(X) = \text{ind}_{\text{CI}_2}(X) \) and also \( \text{coind}_{\text{CI}_2}(Y) = \text{ind}_{\text{CI}_2}(Y) \). Show that \( X \rightarrow Y \) if and only if \( \text{ind}_{\text{CI}_2}(X) \leq \text{ind}_{\text{CI}_2}(Y) \).

Exercise 8 ([5 points]). For a metric CI\(_2\)-space \( X \) show that \( \text{ind}_{\text{CI}_2}(X) \geq n \) if and only if there are closed sets \( A_1, A_2, \ldots, A_{n+1} \subseteq X \) with \( A_i \cap \nu(A_i) = \emptyset \) for all \( i \in [n+1] \) and \( \bigcup_{i \in [n+1]} (A_i \cup \nu(A_i)) = X \).

Exercise 9 ([5 points]). Let \( \mathcal{X} \) be a family of metric CI\(_2\)-spaces with the following two properties:

1. \((S^0, -) \in \mathcal{X}\); (2) If \( X \in \mathcal{X} \) and \( A \) is a closed invariant subset set of \( X \), i.e. \( \nu(A) = A \), then \( A \in \mathcal{X} \).

Let \( I : \mathcal{X} \to \{0, 1, 2, \ldots \} \cup \{\infty\} \) be a function satisfying, for all \( X, Y \in \mathcal{X} \):

(i) If \( X \rightarrow Y \), then \( I(X) \leq I(Y) \).

(ii) If \( X = A \cup B \) for closed invariant sets \( A \) and \( B \), then \( I(X) \leq I(A) + I(B) + 1 \).

(iii) \( I(S^0) = 0 \).

Prove that \( I(X) \leq \text{ind}_{\text{CI}_2}(X) \) for all \( X \in \mathcal{X} \).

HINT: Use exercise 8.