Fair Division aka Cake Cutting

"How to cut a cake fairly?"

- Different notions:
  - proportional: \( \frac{1}{n} \)
  - envy-free:
    - No one else receives a larger piece.
    - "A compromise to the art of dividing a cake such that everyone thinks they received the largest piece."
  - super envy-free:
    - All other players receive at most \( \frac{1}{n} \)

\[ I = \left[ 0, 1 \right] \]

With \( n \) measures:

- E.g.: Moving knife scheme:
  \[ t_0 = \min \left\{ t \mid \exists i : \mu_i \left( t, \bar{t} \right) = \frac{1}{n} \right\} \]

  \[ \Rightarrow \text{ Player } i \text{ gets } \left[ 0, t_0 \right], \]

  \[ \text{ other players are paid with } \left[ t_0, 1 \right]. \]

  \[ \Rightarrow \text{ proportional, but not necessarily envy-free.} \]

- Necklace splitting:
  - 2 thieves
  - \( d \) kinds of stones
  - Can divide with \( d \) cuts.
  - Can also be interpreted as a fair division question:
    - \( d \) opinions on what is valuable (\( d \) measures)
    - \( 0, 0 \) to be divided into 2 pieces, e.t. on \( 0, 1 \)
    - everybody thinks the pieces each have size \( \frac{1}{2} \).
Two siblings inheriting a piece of land, the family also has to think its $\frac{1}{2}, \frac{1}{2}$.

Law of sea treaty...
Brouwer's Fixed Point Theorem & Sperner's Lemma

**THM:**
\[ f : B^n \to B^n \text{ cts } \implies \exists x \in B^n : f(x) = x \]
(Already saw proof using B-U-Thm)

**A Homological Proof:**

Assume \( f : B^n \to B^n \) without fixed point.

Define \( r : B^n \to S^{n-1} : \)

\[ r(x) = \text{point where the ray from } \]
\[ f(x) \text{ through } x \text{ hits } S^{n-1}(= \partial B^n) \]

(And this in previous proof)

Then, \( r \) is continuous \( \Rightarrow r/|S^{n-1}| = \text{id} \),

\[ r \circ i = \text{id}_{|S^{n-1}|} \]

\[ \text{where } i : S^{n-1} \to B^n \text{ inclusion.} \]

Consider induced maps in homology:

\[ \begin{array}{cccc}
\text{Id}^* & H_*(S^{n-1}, \mathbb{Z}_2) & \to & H_*(B^n, \mathbb{Z}_2) \\
\downarrow & \text{id}^* & & \downarrow \\
\mathbb{Z}_2 & 0 & & \mathbb{Z}_2
\end{array} \]
Just as Birkhoff has Tucker as combinatorial equivalent, Brouwer is equivalent to:

**Sperner Lemma** (Z in Exercise)

A subdivision of \( \Delta^n \).

\[ \lambda: V(K) \to \text{Int}(I) \text{ labeling with:} \]

- \( \lambda(e_i) = i \)
- \( \nu \in \text{conv } e_{i_1}, \ldots, e_{i_k} \Rightarrow \lambda(\nu) \in e_{i_{1}, \ldots, i_k} \)

**Then:** A fully-labeled k-simplex in \( \mathbb{R}^{k+1} \).

**Proof with Brouwer:**

A Sperner labeling

**Define:** \( f: \Delta^n \to \Delta^n \)

as affine extension of simplicial map

\[ V(K) \to V(\Delta^n) \]

\[ \nu \mapsto e_{\lambda(\nu)+1} \mod (n+1) \]

Here, \( \Delta_n \) is considered as the standard n-simplex: the convex hull of the standard basis vectors \( e_1, \ldots, e_{n+1} \) in \( \mathbb{R}^{n+1} \)

Even more:

- # fully labeled simplices is odd.

\[ \dim = 3 \]

\[ 1 \to 2 \]

\[ 2 \to 4 \]

\[ 3 \to \text{cycle} \]

\[ 4 \to 1 \]
Now, if fully labeled simplex \( \Rightarrow \) $f$ surjective

\[ \Leftrightarrow \exists x \in \text{interior of } \Delta^n : x \in \text{im } f \]

But: $f$ is fixed-point free on the boundary of $\Delta^n$.

So, the fixed point given by Brouwer has to lie in the interior!