The Szemerédi–Trotter theorem using polynomials

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We apply the Guth–Katz method of “polynomial partitions” in yet another simple proof of the Szemerédi–Trotter theorem on point-line incidences.

1 Preliminaries on polynomials

We will consider mostly bivariate polynomials \( f = f(x, y) = \sum_{i,j} a_{ij} x^i y^j \in \mathbb{R}[x, y] \). The degree of \( f \) is \( \deg(f) = \max\{i + j : a_{ij} \neq 0 \} \). Let \( Z(f) = \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\} \) be the zero set of \( f \).

1.1 Lemma. If \( \ell \) is a line in \( \mathbb{R}^2 \) and \( f \in \mathbb{R}[x, y] \) is of degree at most \( d \), then either \( \ell \subseteq Z(f) \), or \(|\ell \cap Z(f)| \leq d\).

Proof. Writing \( \ell \) in parametric form \( \{(u_1 t + v_1, u_2 t + v_2) : t \in \mathbb{R}\} \), we get that the points of \( \ell \cap Z(f) \) are roots of the univariate polynomial \( g(t) := f(u_1 t + v_1, u_2 t + v_2) \), which is of degree at most \( d \). Thus, either \( g \) is identically 0, or it has at most \( d \) roots.

1.2 Lemma. If \( f \in \mathbb{R}[x, y] \) is nonzero and of degree at most \( d \), then \( Z(f) \) contains at most \( d \) distinct lines.

Proof. We need to know that a nonzero bivariate polynomial (i.e., with at least one nonzero coefficient) does not vanish on all of \( \mathbb{R}^2 \). There are several ways of proving this—we leave it as a challenge for the reader to find one.

Now we fix a point \( p \in \mathbb{R}^2 \) not belonging to \( Z(f) \). Let us suppose \( Z(f) \) contains lines \( \ell_1, \ldots, \ell_k \). We choose another line \( \ell \) passing through \( p \) that is not parallel to any \( \ell_i \) and not passing through any of the intersections \( \ell_i \cap \ell_j \). (Such an \( \ell \) exists since only finitely many directions need to be avoided.) Then \( \ell \) is not contained in \( Z(f) \) and it has \( k \) intersections with \( \bigcup_{i=1}^k \ell_i \). Lemma 1.1 yields \( k \leq d \).

2 The polynomial ham-sandwich theorem

We assume the ham sandwich theorem in the following discrete version: Every \( d \) finite sets \( A_1, \ldots, A_k \subset \mathbb{R}^k \) can be simultaneously bisected by a hyperplane. Here a hyperplane \( h \)
bisects a finite set $A$ if neither of the two open halfspaces bounded by $A$ contains more than $\lfloor |A|/2 \rfloor$ points of $A$.

From this, it is easy to derive the polynomial ham-sandwich theorem (which we state for bivariate polynomials).

2.1 Theorem. Let $A_1, \ldots, A_t \subseteq \mathbb{R}^2$ be finite sets, and let $d$ be an integer with $(d+2) - 1 \geq t$. Then there exists a nonzero polynomial $f \in \mathbb{R}[x, y]$ of degree at most $d$ that simultaneously bisects all the $A_i$, where “$f$ bisects $A_i$” means that $f > 0$ in at most $\lfloor |A_i|/2 \rfloor$ points of $A_i$ and $f < 0$ in at most $\lfloor |A_i|/2 \rfloor$ points of $A_i$.

Proof. We note that $(d+2)$ is the number of monomials in a bivariate polynomial of degree $d$, or in other words, the number of pairs $(i, j)$ of nonnegative integers with $i + j \leq d$. We set $k := (d+2) - 1$, and we let $\Phi: \mathbb{R}^2 \to \mathbb{R}^k$ be the Veronese map given by

$$\Phi(x, y) := (x^i y^j)_{1 \leq i + j \leq d} \in \mathbb{R}^k.$$  

(We think of the coordinates in $\mathbb{R}^k$ as indexed by pairs $(i, j)$ with $1 \leq i + j \leq d$.)

Assuming, as we may, that $t = k$, we set $A'_i := \Phi(A_i)$, $i = 1, 2, \ldots, k$, and we let $h$ be a hyperplane simultaneously bisecting $A'_1, \ldots, A'_k$. Then $h$ can has an equation of the form $a_{00} + \sum_{i,j} a_{ij} z_{ij} = 0$, where $(z_{ij})_{1 \leq i + j \leq d}$ are the coordinates in $\mathbb{R}^k$. It is easy to check that $f(x, y) := \sum_{i,j} a_{ij} x^i y^j$ is the desired polynomial. \qed

3 Proof of the Szemerédi–Trotter theorem

For a finite set $P \subseteq \mathbb{R}^2$ and a finite set $L$ of lines in $\mathbb{R}^2$, let $I(P, L)$ denote the number of incidences of $P$ and $L$, i.e., of pairs $(p, \ell)$ with $p \in P$, $\ell \in L$, and $p \in \ell$.

3.1 Theorem (Szemerédi–Trotter). We have $I(P, L) = O(m^{2/3}n^{2/3} + m + n)$ for every $m$-point $P$ and every set $L$ of $n$ lines.

Let us say that sets $P, Q \subseteq \mathbb{R}^2$ are strictly separated by a polynomial $f \in \mathbb{R}[x, y]$ if $P \cap Z(f) = Q \cap Z(f) = \emptyset$, and every segment $pq$, $p \in P$, $q \in Q$, intersects $Z(f)$.

Let $P \subseteq \mathbb{R}^2$ be an $m$-point set, and let $s > 1$ be a parameter. We say that $f \in \mathbb{R}[x, y]$ is an $s$-partitioning polynomial for $P$ if the set $P \setminus Z(f)$ can be partitioned into disjoint subsets $P_1, \ldots, P_t$ so that $t = O(s)$, $|P_i| \leq m/s$ for all $i$, and for every $i \neq j$, $P_i$ and $P_j$ are strictly separated by $f$.

3.2 Lemma (Polynomial partitioning lemma). For every $s > 1$, every finite $P \subseteq \mathbb{R}^2$ admits an $s$-partitioning polynomial $f$ of degree $O(\sqrt{s})$.

Proof. We inductively construct collections $\mathcal{P}_0, \mathcal{P}_1, \ldots$, each consisting of disjoint subsets of $P$. We start with $\mathcal{P}_0 := \{P\}$. Having constructed $\mathcal{P}_j$ with at most $2^j$ sets, we use the polynomial ham-sandwich theorem to construct a polynomial $f_j$ that bisects each of the sets of $\mathcal{P}_j$. Then for every class $Q \in \mathcal{P}_j$, we let $Q'$ consist of the points of $Q$ on which $f_j > 0$, $Q''$ consists of those points of $Q$ where $f_j < 0$, and $\mathcal{P}_{j+1} := \bigcup_{Q \in \mathcal{P}_j} \{Q', Q''\}$ (empty sets ignored).
The sets in $P_j$ have size at most $|P|/2^j$. We let $k := \lceil \log_2 s \rceil$; then the sets in $P_k$ have size at most $|P|/s$ and they form the desired $P_1, \ldots, P_t$, where $t \leq 2^k \leq 2s$. We also set $f := f_1 f_2 \cdots f_k$.

Then $f$ is an $s$-partitioning polynomial for $P$ by the construction, and it remains to bound $\deg(f)$. By the polynomial ham sandwich theorem, for bisecting the at most $2^j$ sets in $P_j$, a polynomial $f_j$ of degree $O(\sqrt{2^j})$ suffices. Thus, $\deg(f) = O\left( \sum_{j=1}^{k} O(2^j/2) \right) = O(\sqrt{s})$. □

**Proof of the Szemerédi–Trotter theorem.** For simplicity, we prove the theorem for $m = n$. We set $s := n^{2/3}$, and we let $f$ be an $s$-partitioning polynomial for $P$. By the polynomial partitioning lemma, we may assume $r := \deg(f) = O(\sqrt{s})$.

Let $P_1, \ldots, P_t$ be the sets as in the definition of an $s$-partitioning polynomial for $P$, and let $R := P \cap Z(f)$. Further let $L_0 \subset L$ consist of the lines of $L$ contained in $Z(f)$; we have $|L_0| \leq r$ by Lemma 1.2.

We decompose

$$ I(P, L) = \sum_{i=1}^{t} I(P_i, L) + I(R, L_0) + I(R, L \setminus L_0). $$

We can immediately bound

$$ I(R, L_0) \leq |L_0| \cdot |R| \leq |L_0| n \leq nr = O(n^{4/3}), $$

and

$$ I(R, L \setminus L_0) \leq |L \setminus L_0| r = O(n^{4/3}), $$

since each line of $L \setminus L_0$ intersects $Z(f)$, and thus also $R$, in at most $r = \deg(f)$ points.

It remains to bound $\sum_{i=1}^{t} I(P_i, L)$. Let $L_i \subset L$ be the lines containing at least one point of $P_i$ (the $L_i$ are typically not disjoint). By Lemma 1.1, no line intersects more than $r + 1$ of the $P_i$, and so $\sum_{i=1}^{t} |L_i| \leq (r + 1)n$.

Let us further divide $L_i$ into $L_i'$, the lines containing exactly one point of $P_i$, and $L_i''$, the lines containing at least two points of $P_i$.

We have $I(P_i, L_i'') \leq |P_i|^2$, because for every $p \in P_i$, there are at most $|P_i| - 1$ lines that pass through $p$ and contain at least one other point of $P_i$. Obviously, $I(P_i, L_i') \leq |L_i'|$. Thus, we can estimate

$$ \sum_{i=1}^{t} I(P_i, L) = \sum_{i=1}^{t} I(P_i, L_i') + \sum_{i=1}^{t} I(P_i, L_i'') \leq \sum_{i=1}^{t} |L_i'| + \sum_{i=1}^{t} |P_i|^2 $$

$$ \leq (r + 1)n + t \left( \frac{n}{s} \right)^2 = O(n^{4/3}). $$

□