COMPLEXITY OF EQUILIBRIA

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Nash equilibria in bimatrix games

Let $G(A, B)$ be nonsymmetric bimatrix game, where A and B are $m \times n$ payoff matrices. We assume, that G is nondegenerate, that is: the number of pure best responses to any mixed strategy never exceeds the size of its support. Let

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix}.$$

We have a **linear complementarity problem**:
Given C and q ($q = 1$), find z such that

$$z \geq 0, \qquad q - Cz \geq 0, \quad z^T(q - Cz) = 0.$$

Sketch of Lemke-Howson algorithm

We define a polytope

$$S = \{z \in \mathbb{R}^{m+n} | z \geq 0, Cz \leq 1\}$$

which is the product $P \times Q$ of the polytopes

$$P = \{x \in \mathbb{R}^m | x \geq 0, B^T x \leq 1\}, \quad Q = \{y \in \mathbb{R}^n | Ay \leq 1, y \geq 0\}.$$

Any Nash equilibrium is given by $z^T(1 - Cz) = 0$, which is equivalent to $x^T(1 - Ay) = 0$ and $y^T(1 - B^T x) = 0$ (x and y have to be normalized to present mixed strategies). It is easy to see, that $(0, 0)$ is the artificial equilibrium. It would be a starting point for Lemke-Howson algorithm.

For each pure strategy i, the facets of S defined by $z_i = 0$ and by $(Cz)_i = 1$ both get label i. Every vertex is labeled by the facets it lies on. The complementarity condition $z^T(1 - Cz) = 0$ means that z is completely labeled.

The LH algorithm starts from completely labeled vertex $z = 0$ by choosing label k of a facet to be dropped. This is the only free choice of the algorithm. By leaving the facet with label k we select a unique edge, which takes us to new vertex z'. It lies on a new facet labeled by j. If $j = k$ then algorithm stops and z' is an equilibrium, otherwise j is duplicate and the next edge is chosen by leaving the facet that so far has label j, and the process is repeated.

LH on labeled dual cyclic polytopes

A standard way of obtaining a cyclic polytope P' is to take the convex hull of $2d$ points $\mu(t_i)$ on the moment curve $\mu : t \to (t, t^2, \ldots, t^d)$ for $i = 1, \ldots, 2d$ ($t_1 < t_2 < \ldots < t_{2d}$). The dual cyclic polytope is defined as

$$P'' = \{x \in \mathbb{R}^d | c_i^T z \leq 1, 1 \leq i \leq 2d\},$$

where $c_i = \mu(t_i) - \overline{\mu}$.

A vertex u of such polytope is characterized by a bitstring $u_1 u_2 \ldots u_{2d}$, with the kth bit u_k indicating whether u is on the kth facet or not. The polytope is simple, hence exactly d bits are 1.

The bitstrings fulfill the Gale evenness condition: A bitstring represents a vertex if and only if any substring of the form $01 \ldots 10$ has an even length. The only odd runs of 1's that are allowed must be at both ends of string.

Both P and Q will be dual cyclic polytopes with a special order of their inequalities. The equilibrium condition and the LH algorithm depend on which facets a vertex belongs to, as encoded in the Gale evenness bitstrings, and on the facet labels. These are defined by permutations l and l' for P and Q respectively. l is identity permutation and

$$l'(k) = \begin{cases} k, & k = 1, d, \\ k + (-1)^k, & 2 \leq k \leq d-1, \\ k - (-1)^k, & d+1 \leq k \leq 2d. \end{cases}$$

Lemma *The only two completely labeled vertices are* $e_0 = (1^d 0^d, 0^d 1^d)$ *and* $e_1 = (0^d 1^d, 1^d 0^d)$*, where* e_0 *is an artificial equilibrium and* e_1 *is the only Nash equilibrium of the game.*

The LH algorithm proceeds as follows:

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**1**	**3**	**2**	**4**	**6**	**5**	**8**	**7**
$e_0 =$	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1
	1	1	1	0	0	0	0	1	0	0	0	0	1	1	1	1
	1	1	1	0	0	0	0	1	1	0	0	0	1	1	0	1
	0	1	1	0	0	0	1	1	1	0	0	0	1	1	0	1
	0	1	1	0	0	0	1	1	1	1	0	0	1	1	0	0
	1	1	0	0	0	0	1	1	1	1	0	0	1	1	0	0
	1	1	0	0	0	0	1	1	0	1	1	0	1	1	0	0
	1	0	0	0	0	1	1	1	0	1	1	0	1	1	0	0
	1	0	0	0	0	1	1	1	0	1	1	0	0	1	1	0
	1	1	0	0	0	1	1	0	0	1	1	0	0	1	1	0
	1	1	0	0	0	1	1	0	1	1	0	0	0	1	1	0
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$e_1 =$	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0

Theorem **(Savani, von Stengel)** *There are* $d \times d$ *games, for even* d*, where each LH path has length* $\Omega(\theta^{3d/4})$*.*

Sketch of the proof:

By $\pi(d, k)$ we mean the path of the algorithm when label k is dropped in dimension d of the game. Similarly by $L(d, k)$ we denote length of such path.

Lemma *Let* $A(d) = \pi(d, 1)$ *and* $B(d) = (u^1, v^1) \ldots (u^{L-2}, v^{L-2})$*, where* (u^i, v^i) *is the* i*th vertex pair of* $\pi(d, 2d)$*,* $0 \le i \le L = L(d, 2d)$*. Then there are paths* $C(d)$ *and mappings* α*,* β*,* β'*,* γ*,* γ' *defined on vertex pairs, and extended to sequences of vertex pairs, such that*

$$\begin{aligned} A(d) &= \beta(B(d)) + C(d), & (1) \\ C(d) &= \alpha(A(d-2)) + \beta'(B(d)), & (2) \\ B(d) &= \gamma(A(d-2)) + \gamma'(C(d-2)). & (3) \end{aligned}$$

Let a_n be the number of vertex pairs of $A(2n)$, which is one more than the length $L(2n,1)$ of that path. Let b_n and c_n be that number for $B(2n)$ and $C(2n)$, respectively. That is,

$$a_n = L(2n,1) + 1, \qquad b_n = L(2n,4n) - 2 \qquad (n \geq 1). \tag{4}$$

Then the (1) implies $a_n = b_n + c_n$, (2) — $c_n = a_{n-1} + b_n$, and (3) — $b_n = a_{n-1} + c_{n-1}$. Moreover, the paths $\pi(2,1)$ and $\pi(2,4)$ have length $4 = a_1 - 1 = b_1 + 2$. This shows that the numbers $b_1, c_1, a_1, b_2, c_2, a_2, \ldots$ are the Fibonacci numbers $2, 3, 5, 8, 13, 21, \ldots$ given by

$$f_0 = 1, \qquad f_1 = 2, \qquad f_{n+1} = f_n + f_{n-1}, \qquad (n \geq 1),$$

that is,

$$a_n = f_{3n}, \qquad b_n = f_{3n-2}, \qquad (n \geq 1). \tag{5}$$

Lemma *The LH path lengths for any dropped label are characterized by (4) and (5), and*

(a) $L(d,k) = L(d,d+1-k)$ *and* $L(d,d+k) = L(d,2d+1-k)$*, for* $1 \le k \le d$*,*

(b) $L(d,k) = L(d,k+1)$ *for even* k *when* $2 \le k \le d-2$*, and odd* k *when* $d+1 \le k \le 2d-1$*,*

(c) $L(d,k) = L(k,1) + L(d-k,1)$ *for even* k *and* $2 \le k \le d-2$*,*

(d) $L(d,d+k) = L(k,2k) + L(d-k+2,2(d-k+2)) - 4 = b_{k/2} + b_{d/2-k/2+1}$ *when* k *is even and* $4 \le k \le d-2$*.*

The shortest path lengths are in general obtained when dropping label $3d/2$.

$$L(d,3d/2) = b_{d/2} + b_{d/2+1} = b_{d/2} + a_{d/2} + c_{d/2} = 2a_{d/2}.$$

The Fibonacci numbers are given by the well-known explicit expression

$$f_n = K\theta^n + \overline{K\theta}^n, \quad \theta, \overline{\theta} = 0.5 \pm 0.5\sqrt{5}, \quad K, \overline{K} = 0.5 \pm 0.3\sqrt{5},$$

where $\theta = 1.618\ldots$ is the Golden Ratio. By case (d) of the last Lemma, the sequence of shortest LH path lengths $L(2n, 3n)$ for $n = d/2 = 1, 2, 3, \ldots$ is 4, 10, 16, 42, 68, 178, $\ldots$, which is the sequence of Fibonacci numbers (multiplied by two) with every third number omitted. These shortest lengths grow with the square root of the longest lengths, which is still exponential.

Construction of the game

Dual cyclic polytope in $\mathbb{R}^d$ with $2d$ facets is obtained from the moment curve $\mu : t \rightarrow (t, t^2, t^3, \ldots, t^d)^T$. Let $t_1 < t_2 < \ldots < t_{2d}$ and $c_i = \mu(t_i) - \overline{\mu}$. Then

$$P' = \{z \in \mathbb{R}^d | c_i^T z \leq 1, 1 \leq i \leq 2d\}$$

defines the dual cyclic polytope.

The affine transformation of P' is given by

$$P = \{x \in \mathbb{R}^d | x \geq 0, -DC^{-1}x \leq r\},$$

where $Cz \leq 1$ represents the first d inequalities in P', $Dz \leq 1$ the last d inequalities, and $r = 1 - DC^{-1}1$.

Let S be the diagonal matrix with $s_{ii} = 1/r_i$ and $s_{ij} = 0$ for $i \neq j$. Then

$$P = \{x \in \mathbb{R}^d | x \geq 0, -SDC^{-1}x \leq 1\}.$$

is a cyclic polytope with facets characterized by Gale evenness strings.

Theorem (Savani, von Stengel) *A* $d \times d$ *bimatrix game* $G(A, B)$*, for which LH algorithm paths have exponential length is given by* $B^T = -SDC^{-1}$*. The matrix* A *is obtained from* B *by*

$$a(l'(i), l'(j+d)-d) = b(j,i), \qquad (1 \le i, j \le d),$$

where

$$l'(k) = \begin{cases} k, & k = 1, d, \\ k + (-1)^k, & 2 \le k \le d-1, \\ k - (-1)^k, & d+1 \le k \le 2d. \end{cases}$$

General complexity results

Theorem (Nash 1950) *In every finite game there exists a NE.*

How complex is it to construct such an equilibrium?

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How complex is it to construct such an equilibrium?

- No general result available.
- There are some related questions that we can answer:
  1. How complex is it to compute all Nash equilibria?
  2. How complex is it to find "good" Nash equilibrium?
  3. How complex is it to compute the number of all the Nash equilibra?

Theorem *Even in symmetric 2-player games, it is NP-hard to determine whether there exists a NE with expected social welfare at least k (even if k is the highest social welfare that can be obtained in the game).*

Theorem *Even in symmetric 2-player games, it is NP-hard to determine whether there exists a NE where all players have expected utility at least k.*

Theorem *Even in symmetric 2-player games, it is NP-hard to determine whether there exists a Pareto-optimal NE.*

Theorem *Even in symmetric 2-player games, it is NP-hard to determine whether there exists a NE where player 1 has expected utility of at least k.*

Definition *Let ϕ be a boolean formula in conjunctive normal form. Let V be its set of variables, L — the set of positive and negative literals corresponding to these variables, C — the set of its clauses. Let game $G(\phi)$ be defined as follows:*

1. *The sets of actions* $\Sigma = \Sigma_1 = \Sigma_2 = L \cup V \cup C \cup \{f\}$.
2. *The utility functions* u_1 *and* u_2 *are:*
   - $u_1(l^1, l^2) = 1$ *for all* $l^1, l^2 \in L$ *with* $l^1 \neq -l^2$;
   - $u_1(l, -l) = -2$ *for all* $l \in L$;
   - $u_1(l, x) = -2$ *for all* $l \in L$, $x \in \Sigma - L$;
   - $u_1(v, l) = 2$ *for all* $v \in V$, $l \in L$ *with* $v(l) \neq v$;
   - $u_1(v, l) = 2 - n$ *for all* $v \in V$, $l \in L$ *with* $v(l) = v$;
   - $u_1(v, x) = -2$ *for all* $v \in V$, $x \in \Sigma - L$;
   - $u_1(c, l) = 2$ *for all* $c \in C$, $l \in L$ *with* $l \notin c$;
   - $u_1(c, l) = 2 - n$ *for all* $c \in C$, $l \in L$ *with* $l \in c$;
   - $u_1(c, x) = -2$ *for all* $c \in C$, $x \in \Sigma - L$;
   - $u_1(f, f) = 0$;
   - $u_1(f, x) = 1$ *for all* $x \in \Sigma - \{f\}$.

   $u_2(x, y) = u_1(y, x)$ *for all* $x, y \in \Sigma$.
   Function $v : L \to V$ *gives the variable corresponding to a literal e.g.* $v(x_1) = v(-x_1) = x_1$.

Theorem *If* $(l_1, l_2, \ldots, l_n)$ *(where* $v(l_i) = x_i$*) satisfies* ϕ*, then there is a Nash equilibrium in* $G(\phi)$ *where both players play* l_i *with probability* $\frac{1}{n}$*, with expected utility* 1 *for each player. The only other Nash equilibrium is the one where both players play* f*, and receive expected utility* 0 *each.*

CNF SAT:

Given a boolean formula in conjunctive normal form e.g.

$$\exists x_1 \exists x_2 \exists x_3 \exists x_4 (x_1 \vee \neg x_3 \vee x_4) \wedge (\neg x_2 \vee x_3),$$

is there an assignment of **TRUE** and **FALSE** values to the variables that will make the entire expression true?

CNF SAT is NP-complete

Proof of first of complexity theorems:

- By theorem in every game $G(\phi)$ there is an equilibrium with social welfare 0 and there may be an equilibrium with social welfare 2 only if there exists $(l_1, l_2, \ldots, l_n)$ satisfying ϕ.
- $\Rightarrow$ Our problem is NP-hard.

Theorem *Even in symmetric 2-player games, counting the number of Nash equilibria is #P-hard.*

Proof:

1. Counting the number of satisfying assignments to a CNF formula is #P-hard.
2. The number of Nash equilibria in game $G(\phi)$ is one plus the number of satisfying assignments to the variables of ϕ.
   $\Rightarrow$ Counting the number of Nash equilibria must also be #P-hard.

Complexity of finding pure equilibria

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- What happens in games with more complex strategy spaces?

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- What happens in games with more complex strategy spaces?

Theorem *Even in symmetric 2-person Bayesian games where prior distribution over the set of types is uniform, determining whether there exists a pure-strategy Bayes-Nash equilibrium is NP-hard.*

Proof: A reduction from SET-COVER problem.

> SET-COVER: For a given finite set S, its subsets S_1, S_2, . . . , S_m, such that $S = \bigcup_{i=1}^{m} S_i$ and integer $k < m$, do there exist S_{c_1}, . . . , S_{c_k} such that $S = \bigcup_{i=1}^{k} S_{c_k}$.
> SET-COVER is NP-complete.

Definition *A discounted stochastic game is defined by:*

- *a set of players* A*;*
- *a set of states* S*, among which the game transits;*
- *for each player* i*, a set of actions* Σ_i *that can be played in any state;*
- *a transition probability* $q : S \times \Sigma_1 \times \cdots \times \Sigma_{|A|} \times S \to [0,1]$*, where* $q(s_1, a_1, \ldots, a_{|A|} \mid s_2)$ *gives the probability of the game being in state* s_2 *in the next stage given that the current state is* s_1 *and players play actions* a_1*, . . . ,* $a_{|A|}$*;*
- *for each player* i *a payoff function* $u_i : S \times \Sigma_1 \times \cdots \times \Sigma_{|A|} \to \mathbb{R}$*, where* $u_i(s, a_1, \ldots, a_{|A|})$ *gives payoff to player* i *in state* s *if players play actions* $a_1, \ldots, a_{|A|}$*.*

  The total payoff of player i *is*

$$\sum_{k=0}^{\infty} \beta^k u_i(s^k, a_1^k, \ldots, a_{|A|}^k),$$

where s^k *is the state of the game at stage* k*, while* $a_1^k, \ldots, a_{|A|}^k$ *are actions of the players at stage* k*.*

We say that a stochastic game is called invisible when the players are not aware neither of the actions of the other players nor of the payoffs acumulated by any of the players.

Theorem *Even in a symmetric 2-player invisible stochastic game with deterministic transitions it is PSPACE-hard to find a pure-strategy Nash equilibrium.*

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- Proof — reduction from PERIODIC-SAT (PSPACE-complete)
- Similar result holds for stochastic games with finite number of stages.

The review of positive results for particular classes of games

Correlated equilibria

Definition *Let* $G = (\{S_i\}, \{u_i\})$ *be an* n*-person game. Let* q *be a probability distribution on* $S_1 \times \ldots \times S_n$*. Distribution* q *is a correlated equilibrium if for each player* i *and each pair* l*,* l' *of strategies in* S_i*,*

$$\sum_{s:s_i=l} q(s)u_i(s) \geq \sum_{s:s_i=l} q(s)u_i(s'),$$

where $s' = (s_{-i}, l')$*.*

An interpretation is as follows. All players together choose a probability distribution q and give it to some arbiter. Arbiter picks a strategy profile s at random according to q and recommends strategy s_i to each player i (s_i and q are the only information available for player i). If noone can make profit on changing s_i to some other strategy s_i', for every possible (with respect to q) choice of s, then q is a correlated equilibrium.

Exponential input complexity

To represent a game in normal form, we need nk^n numbers, where n is the number of players and k is the number of pure strategies available to each player. For some games it is possible to decrease it.

Symmetric games

A game is symmetric if $S_1 = \ldots = S_n$, $u_i(s)$ depends only on s_i and the other players' strategies (but not on i), and $u_i(s)$ is a symmetric function of s_{-i}. In other words, the payoff to player depends only on his strategy and on the number of players choosing each of the strategies. Can be specified by giving only $k\binom{n+k-1}{k-1}$ numbers.

Graphical games

The players are the vertices of a graph, and the payoff of each player depends only on his strategy and those of his neighbors. The number of parameters is exponential in the maximum degree but polynomial in the numbers of players.

Symmetric games (with two strategies)

Let $G = (S = \{1, 2\}, u_1, \ldots, u_n)$. Let $p_i(j)$ be the aggregate probability assigned to the strategy profiles $S_i(j)$ in which exactly j players, including player i, choose strategy 1. Let $p(j)$ be the total probability of the strategy profiles $S(j)$ in which exactly j players choose strategy 1. Let us consider the following basic linear system $L(G)$:

$$
\begin{aligned}
\sum_{j=0}^{n} p_i(j) u_i(j, 1) &\geq \sum_{j=0}^{n} p_i(j) u_i(j-1, 2) \\
\sum_{j=0}^{n} [p(j) - p_i(j)] u_i(j, 2) &\geq \sum_{j=0}^{n} [p(j) - p_i(j)] u_i(j+1, 1) \\
\sum_{j=0}^{n} p(j) = 1 & \qquad \sum_{i=0}^{n} p_i(j) = j p(j) \\
0 \leq p_i(j) &\leq p(j) \leq 1,
\end{aligned}
$$

where $u_i(j, l)$ denotes the payoff to player i in a situation when player i chooses strategy l and a total of j players choose strategy 1.

Theorem (Papadimitriou, Roughgarden)
Let G be a 2-strategy symmetric game. Then every solution of basic linear system $L(G)$ can be extended to a correlated equilibrium of G.

General Compact Games

Definition *Let $G = (S_1, \ldots, S_n, u_1, \ldots, u_n)$ be a game in normal form. For $i = 1, 2, \ldots, n$ let $P_i = \{P_i^1, P_i^2, \ldots, P_i^{m_i}\}$ be a partition of S_{-i} into m_i classes.*

1. *For a player i, two strategy profiles s and s' are i-equivalent if $s_i = s'_i$, and both s_{-i} and s'_{-i} belong to the same class of the partition P_i.*
2. *The set $P = \{P_1, \ldots, P_n\}$ of partitions is a compact representation of G if $u_i(s) = u_i(s')$ whenever s and s' are i-equivalent.*

Definition *Let $P = \{P_i^j\}$ be a compact representation of a game G. The separation problem for P is the following algorithmic problem: Given rational numbers $y_i(j, l)$ for all i, j and $l \in S_i$, is there a strategy profile s with $\sum_{(i,j,l):s_i=l,s_{-i}\in P_i^j} y_i(j, l) < 0$?*

Theorem (Papadimitriou, Roughgarden)
Let P be a compact representation of a game G. If the separation problem for P can be solved in polynomial time, then a correlated equilibrium of G can be computed in time polynomial in the size of P.

Corollary *A correlated equilibrium of a symmetric game can be found in time polynomial in its natural compact representation.*

Corollary *A correlated equilibrium of a graphical game with a tree topology can be found in time polynomial in its natural compact representation.*

Results for congestion games

Theorem (Rosenthal 1973) *Every congestion game has a pure Nash equilibrium.*

It is known from the proof of Rosenthal's theorem, that for every congestion game there exists a potential function i.e. function ϕ from $S = S_1 \times \cdots \times S_n$ (where S_j is player j's set of actions) to $\mathbb{R}$, such that for two (aggregate) actions of the players which differ only in i-th component, s and s',

$$\phi(s') - \phi(s) = u_i(s) - u_i(s').$$

Since this kind of function exists, to establish the existence of a Nash equilibrium in a given congestion game it is enough to find a local minimum of ϕ.

Definition *A problem in PLS is given by an input from a finite set of solutions and a polynomial-time algorithm that computes the cost for each solution, and a neighboring solution of lower cost provided that one exists.*

An instance of a PLS problem is: starting from some feasible solution find another feasible solution that is a local minimum of the cost function.

Theorem *It is PLS-complete to find a pure Nash equilibrium in congestion games.*

Sketch of the proof:

1. **POSNAE3FLIP:**
   Given an instance of not-all-equal-3SAT with weights on its clauses and containing positive literals only, find a truth assignment satisfying clauses whose total weight cannot be improved by flipping variable:
   This problem is PLS-complete.

2. **Construction of the game:**

   - Players are variables.
   - For each 3-clause c of weight w we have 2 resources e_c and e'_c with delays 0 if 2 or less players use them, and w otherwise.
   - Player x has 2 strategies:
     use all e_c-s for clauses containing x;
     use all e'_c-s for clauses containing x.

   Any Nash equilibrium of the congestion game corresponds to a local optimum of POSNAE3FLIP instance.

Definition *An n-person network congestion game is given by a network (V, E), two nodes a_i, b_i for each player and a* delay function $d : E \times \{1, \ldots, n\} \to \mathbb{R}$. *$d_e(j)$ is assumed to be nondecreasing in j.*

The actions available to player i are all paths from a_i to b_i (the set of actions available to i will be denoted by S_i), while for every $s = (s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$, the payoff to player i is defined as $u_i(s) = -\sum_{e \in s_i} d_e(|\{j : e \in s_j\}|)$.

Definition *A network congestion game is called symmetric if endpoints a and b are the same for all of the players.*

Theorem *There is a polynomial algorithm for finding a pure Nash equilibrium in symmetric network congestion games.*

The algorithm computes global minimum of ϕ, which is also a local minimum, and hence a Nash equilibrium.

- Given network (V, E, a, b) and delay functions d_e, we replace each edge e in N with parallel edges between the same nodes, each with capacity 1, and costs $d_e(1)$, $d_e(2)$,. . . , $d_e(n)$.
- ϕ can be given by the formula:

$$\phi(s) = \sum_e \sum_{j=1}^{|\{k:e\in s_k\}|} d_e(j).$$

Any integer min-cost flow in the new network minimizes $\phi(s)$.

Theorem *It is PLS-complete to find a pure Nash equilibrium in congestion games. It is still PLS-complete if we restrict our attention to nonsymmetric network congestion games.*

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