Congestion and crowding Games

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Overview

- Congestion and Potential games (Carmine)
- Crowding games (Pasquale)
- Complexity of pure equilibria (Vincenzo)

What is a game?

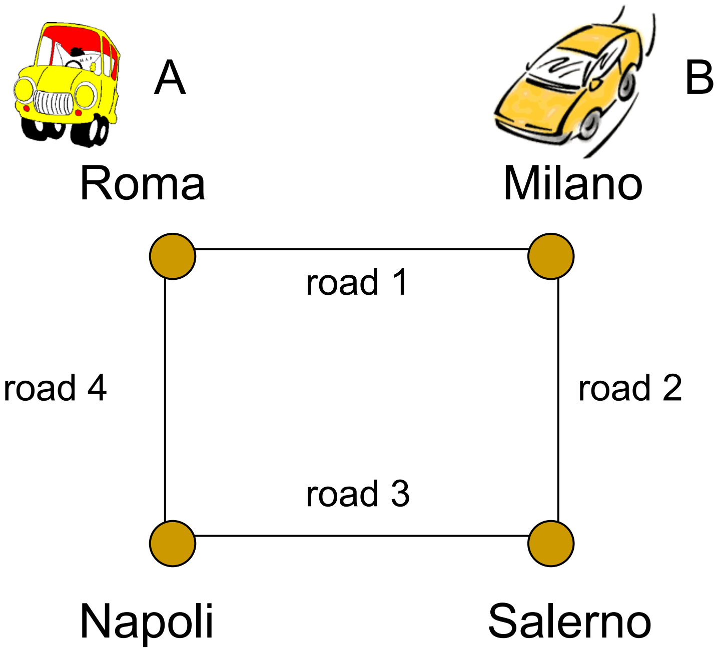
- Classical example: BoS game

		player 2	
		Bach	Stravinsky
player 1	Bach	(2,1)	(0,0)
	Stravinsky	(0,0)	(1,2)

A strategy s (e.g. (B,B)) is a Nash equilibrium iff for all players i it holds:
$u_i(s_{-i},s) >= u_i(s_{-i},s')$
for each s' in S_i

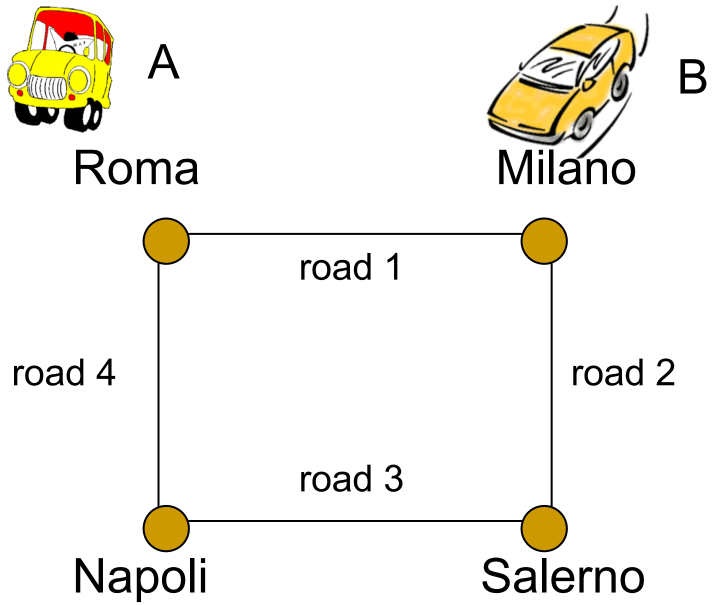
- Players
- Strategies: $S_1 = S_2 = \{\text{Bach, Stravinsky}\}$
- Payoff functions: $u_1(B,B) = 2$, $u_2(B,S) = 0$, …
- Equilibria: i.e. (B,B) and (S,S) are Nash equilibria

(General) Congestion Games



- resources
  - roads 1,2,3,4
- players
  - driver A, driver B
- strategies: which roads I use for reach my destination?
  - A wants to go in Salerno
  - e.g. S_A={{1,2},{3,4}}
  - B wants to go in Napoli
  - e.g. S_B={{1,4},{2,3}}
- what about the payoffs?

Payoffs in (G)CG: an example



SIRF (Small Index Road First)

- A choose path 1,2
- B choose path 1,4
- $u_A = -(c_1(2) + c_2(1)) = -4$
- $u_B = -(c_1(2) + c_4(1)) = -5$

Costs for the roads

$c_1(1)=2$	$c_1(2)= 3$
$c_2(1)=1$	$c_2(2)= 4$
$c_3(1)=4$	$c_3(2)= 6$
$c_4(1)=2$	$c_4(2)= 5$

B

A	{1,4}	{2,3}
{1,2}	(-4,-5)	(-6,-8)
{3,4}	(-9,-7)	(-8,-7)

Payoffs in (G)CG

- The payoff of i depends by congestion of the selected resources
  - u_i is the opposite of the total congestion cost paid by i (C_i)
- For each resource there is a congestion cost (or delay) c_k
- c_k is function of $n_k(s)$ (the number of players that in the state s choose the resource k)
- Therefore the payoff of i in the state s is:

$$u_i(s) = -C_i(s) = -\sum_{k \in s_i} c_k(n_k(s))$$

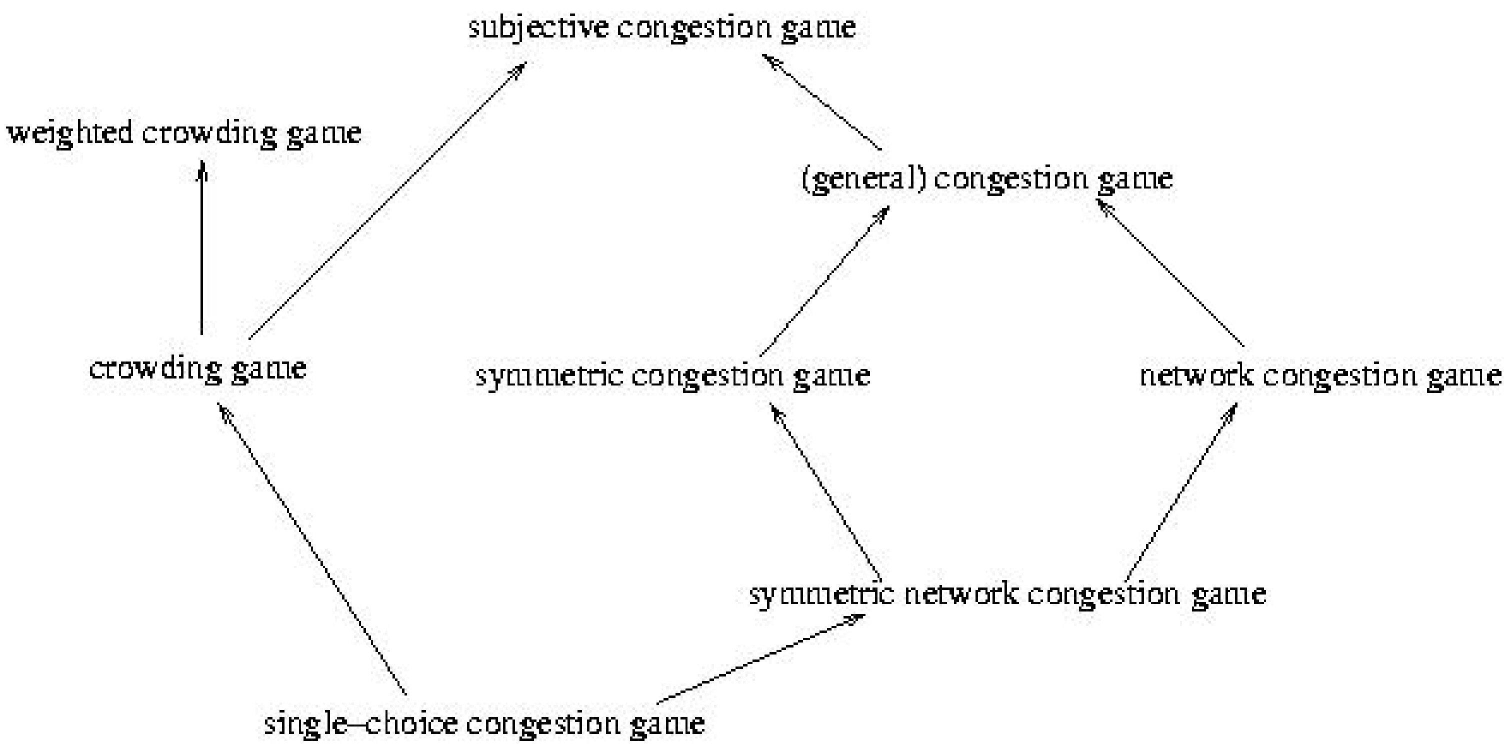
Congestion games: various models

- Symmetric CG
  - S_i are all the same and payoffs are identical symmetric function of n-1 variables
- Single-choice CG
  - Each player can choose only one resource (anyone in the resources set E)
  - Unification of concepts of strategies and resources
- Subjective CG
  - Each player has a different experience of the congestion
  - As consequence every player has a specific payoff

Congestion games: various models (2)

- Network CG
  - Each player has a starting and terminal node and the strategies are the paths in the network
- Crowding game
  - Single-choice subjective CG with payoff non-increasing in $n_k(s)$
- Weighted crowding game
  - Each player has a different *weight* upon the congestion

CGs: relations scheme



GCG and pure Nash equilibria

- Every game has at least one mixed Nash equilibrium
- A game with pure equilibria is “better” than another one with just mixed equilibria

Thm (Rosenthal, 1973)

Every (general) CG possesses at least one pure Nash equilibrium.

Rosenthal’s result

- The class of GCG is “nice”
- We know that there is a class of game for which it is possible to find a pure equilibrium (algorithm?)
- Introduce this (potential) function:

$$P(s) = \sum_{k=1}^{r} \sum_{y=1}^{n_k(s)} c_k(y)$$

Potential functions

- A potential function can trace the "global payoff" of the system along the Nash dynamics
- Several kind of potential functions:
  - Ordinal potential function
  - Weighted potential function
  - (Exact) potential function
  - Generalized ordinal potential function

Ordinal potential function

A function P (from S to R) is an OPF for a game G if for every player i

$$u_i(s_{-i}, x) - u_i(s_{-i}, z) > 0 \text{ iff } P(s_{-i}, x) - P(s_{-i}, z) > 0$$

for every x, z in S_i and for every s_{-i} in S_{-i}

BoS =

(2,1)	(0,0)
(0,0)	(1,2)

P_1 =

4	0
0	2

$u_1(B,B) – u_1(S,B) > 0$ implies that $P_1(B,B) – P_1(S,B) > 0$

$u_1(B,S) – u_1(S,S) < 0$ implies that $P_1(B,S) – P_1(S,S) < 0$

$P_1(B,B) – P_1(S,B) > 0$ implies that $u_1(B,B) – u_1(S,B) > 0$ and that $u_2(B,B) – u_2(S,B) > 0$

P_1 is an ordinal potential function for BoS game

Weighted potential function

A function P (from S to R) is a w-PF for a game G if for every player i

$$u_i(s_{-i}, x) - u_i(s_{-i}, z) = w_i\,(P(s_{-i}, x) - P(s_{-i}, z))$$

for every x, z in S_i and for every s_{-i} in S_{-i}

PD =

	C	D
C	(1,1)	(9,0)
D	(0,9)	(6,6)

P_2 =

2	3/2
3/2	0

$u_1(C,C) – u_1(D,C) = 1 = 2\,(P_2(C,C) – P_2(D,C))$
$u_1(C,D) – u_1(D,D) = 3 = 2\,(P_2(C,D) – P_2(D,D))$

$u_2(D,C) – u_2(D,D) = 3 = 2\,(P_2(D,C) – P_2(D,D))$
$u_2(C,C) – u_2(C,D) = 1 = 2\,(P_2(C,C) – P_2(C,D))$

P_2 is a (2,2)-potential function for PD game

Weighted potential function

A function P (from S to R) is a w-PF for a game G if for every player i

$$u_i(s_{-i}, x) - u_i(s_{-i}, z) = w_i\,(P(s_{-i}, x) - P(s_{-i}, z))$$

for every x, z in S_i and for every s_{-i} in S_{-i}

G' =

	A	B
A	(3,1)	(4,0)
B	(2,4)	(1,0)

P_3 =

11	9
8	0

$u_1(A,A) - u_1(B,A) = 3 - 2 = 1/3\,(P_3(A,A) - P_3(B,A))$
$u_1(A,B) - u_1(B,B) = 4 - 1 = 1/3\,(P_3(A,B) - P_3(B,B))$

$u_2(B,A) - u_2(B,B) = 4 - 0 = 1/2\,(P_3(B,A) - P_3(B,B))$
$u_2(A,A) - u_2(A,B) = 1 - 0 = 1/2\,(P_3(A,A) - P_3(A,B))$

P_3 is a (1/3,1/2)-potential function for the game G'

(Exact) potential function

A function P (from S to R) is an (exact) PF for a game G if it is a w-potential function for G with w_i = 1 for every i

PD =

	C	D
C	(1,1)	(9,0)
D	(0,9)	(6,6)

P_4 =

4	3
3	0

$u_1(C,C) - u_1(D,C) = P_4(C,C) - P_4(D,C)$
$u_1(C,D) - u_1(D,D) = P_4(C,D) - P_4(D,D)$

$u_2(D,C) - u_2(D,D) = P_4(D,C) - P_4(D,D)$
$u_2(C,C) - u_2(C,D) = P_4(C,C) - P_4(C,D)$

P_4 is a potential function for PD game

Generalized ordinal potential function

A function P (from S to R) is an GOPF for a game G if for every player i

$u_i(s_{-i}, x) - u_i(s_{-i}, z) > 0$ implies $P(s_{-i}, x) - P(s_{-i}, z) > 0$

for every x, z in S_i and for every s_{-i} in S_{-i}

G'' =

	A	B
A	(1,0)	(2,0)
B	(2,0)	(0,1)

P_5 =

0	3
1	2

P_5 is a generalized ordinal potential function for the game G''

P_5 is not an ordinal potential function for the game G''

$P_5(A,B) – P_5(A,A) > 0$ implies that $u_1(A,B) – u_1(A,A) > 0$ but not that $u_2(A,B) – u_2(A,A) > 0$

Potential games

- A game that admits an OPF is called an ordinal potential game
- A game that admits a weighted PF is called a weighted potential game
- A game that admits a PF is called a potential game
- Using the potential functions properties we obtain several interesting results
  - E.g., in such games find an equilibrium is equivalent to maximize the potential

Equilibria in Potential Games

Thm (MS96)

Let G be an ordinal potential game (P is an OPF). A strategy profile s in S is a pure equilibrium point for G iff for every player i it holds

$$P(s) >= P(s_{-i}, x) \text{ for every } x \text{ in } S_i$$

Therefore, if P has maximal value in S, then G has a pure Nash equilibrium.

Corollary Every finite OP game has a pure Nash equilibrium.

An example

Nash equilibrium

PD =

	C	D
C	(1,1)	(9,0)
D	(0,9)	(6,6)

P_4 maximal value

P_4 =

4	3
3	0

Thm (MS96)

$PD(P_4)$ =

	C	D
C	(4,4)	(3,3)
D	(3,3)	(0,0)

FIP: an important property

- A path in S is a sequence of states s.t. between every consecutive pair of states there is only one deviator
- A path is an improvement path w.r.t. G if each deviator has a sharp advantage moving

$$u_i(s^k) > u_i(s^{k-1})$$

- G has the FIP if every improvement path is finite
- Clearly if G has the FIP then G has at least a pure equilibrium
  - Every improvement path terminates in an equilibrium point

FIP: an important property (2)

Lemma Every finite OP game has the FIP.

The converse is true? "No"

G'' =

	A	B
A	(1,0)	(2,0)
B	(2,0)	(0,1)

- G'' has the FIP ((B,A) is an equilibrium)
- any OPF must satisfies the following impossible relations:

  P(A,A) < P(B,A) < P(B,B) < P(A,B) = P(A,A)

Lemma Let G be a finite game. Then, G has the FIP iff G has a generalized ordinal potential function.

Congestion vs Potential Games

- Rosenthal states that Congestion games always admit pure Nash equilibria
- MS96's work shows that potential games always admit pure Nash equilibria
- What is the relation?

Thm Every congestion game is a potential game.

Thm Every finite potential game is isomorphic to a congestion game.

Overview

- Congestion and Potential games (Carmine)
- Crowding games (Pasquale)
- Complexity of pure equilibria (Vincenzo)

Crowding Game

- The Rosenthal model supposes that the *payoff* associated with each primary factor is a function of the number of players who include it in their choice.

- Milchtaich**[Mil96]** supposes that the *payoff* function associated with each primary factor is not universal but player specific.

Crowding game

This *generalization* is accompanied assuming two restrictions:

1. *each player chooses only one primary factor*
2. *the payoff received decreases with the number of other players selecting the same primary factor*

Even with this restriction such games have realizations in economics, traffic flow and ecology

Finite Improvement Property

- The existence of exact potential function implies the ***Finite Improvement Property*** **(FIP):** every improvement path is finite.

- Obviously any maximal Finite Improvement Path ends with an equilibrium.

The two-strategies case

Thm[Mil96]: *Crowding games involving only two strategies possess the FIP.*

If a game has not the FIP it may still have a Nash Equilibrium.

[Mil96] provides an example of 2-player game with 3 strategies without the FIP.

Best Reply Paths

- A path in which each deviator shifts to the *best reply* against the strategies played by other players is called **Best Reply Path**.

- The **Finite improvement property (FIP)** implies the **Finite Best Reply Property (FBRP)** but the converse is not true.

Infinite best-reply improvement path

Claim: *IBRP require at least 3 players*.

Proof:

Assume by contrary that 2 players suffice.

When player A shifts strategy, the second player B is negatively effected only if A plays the same strategy as B (congestion).

It is this second player B which makes the next move, thus only possibly increasing the payoff of the player A (monotonicity), thus the strategy played by A remains a Best Reply strategy and Equilibrium is reached.

An infinite best reply improvement path with 3 strategies and 3 players

	Resource 1	Resource 2	Resource 3
Strategy Profile	2 3	1	
Player 1	2 3		1
$u_1^2(1) < u_1^3(1)$			
Player 2	3		1 2
$u_2^1(2) < u_2^3(2)$			
Player 3		3	1 2
$u_3^1(1) < u_3^2(1)$			
Player 1		1 3	2
$u_1^3(2) < u_1^2(2)$			
Player 2	2	1 3	
$u_2^3(1) < u_2^1(1)$			
Player 3	2 3	1	
$u_3^2(2) < u_3^1(2)$			

An infinite best reply improvement path with 3 strategies and 3 players

	Player 1	Player 2	Player 3
Best Payoff	$u_1^3(1)$	$u_2^1(1)$	$u_3^2(1)$
	$u_1^2(1)$	$u_2^3(1)$	$u_3^1(1)$
	$u_1^2(1)$	$u_2^3(2)$	$u_3^1(2)$
	$u_1^3(2)$	$u_2^1(2)$	$u_3^2(2)$
Worst Payoff	$u_1^1(n_1)$	$u_2^2(n_2)$	$u_3^3(n_3)$

The Strategy-tuples (3,1,2) and (2,3,1) are equilibria of this game

The existence of a pure strategy Nash equilibrium

Thm[Mil96]: *Every (unweighted) crowding game possesses a Nash equilibrium in pure strategies.*

Proof schema: By induction on the number of players *n*, by reducing an *n* player game to an *n-1* player game.

Based on two lemma.

Lemma 1

Lemma: *If $j(0),\ldots,j(M)$ is a sequence of strategies, $s^0,s^1,\ldots,s^M$ is a best reply improvement path and s^k result from the deviation of one player from $j(k-1)$ to $j(k)$ $(k=1,2,\ldots,M)$ then $M \leq N$.*

This Lemma is concerned with paths where each deviator moves to the next deviator's present position

Lemma 2

Lemma: *If the deviation at step k is from j(k) to j(k-1) (k=1,2,…,M) then* $M \leq N \cdot (r-1)$

This Lemma is concerned with paths were each deviator takes the last deviator's previous position.

Convergence to an Equilibrium

The proof of the last theorem is constructive. It gives an algorithm that, adding player by player, finds a NE in at most $\binom{n+1}{2}$ steps.

What about reaching a NE "in the game"?

Thm[Mil96] *Given an arbitrary strategy tuple s^1 in a crowding game there exists a best reply improvement path $s^1,\ldots,s^L$ such that s^L is an equilibrium and*

$$L \leq r\binom{n+1}{2}$$

Weighted crowding game

Up till now the players had similar influence upon the congestion. This model is generalized by introducing weights $\beta_1, \beta_2, \ldots, \beta_n$ and modifying the congestion vector

$$n_j = \sum_{i \mid s_i = j} \beta_i \qquad j = 1, 2, \ldots, r$$

Weighted crowding games

- Weighted crowding games involving only *two players* or involving *two strategies* or with players having *equal payoff functions* possess the finite improvement property or (at least) the finite best reply property.

- Therefore these games possess a Nash equilibrium in pure strategies, and the equilibrium can be reached by constructing a maximal best-reply improvement path

Weighted crowding game: the general case

Thm[Mil96]: *Weighted crowding games may not possess a pure strategy Nash equilibrium.*

Even a three-player, three-strategy weighted crowding game may not possess a pure strategy Nash equilibrium.

	Strategy 1	Strategy 2	Strategy 3
Player 1	$u_1^1(n_1) = -10$	$u_1^2(n_2) = 1/15n_2 + 1/4$	$u_1^3(n_3) = 1/n_3$
Player 2	$u_2^1(n_1) = -10$	$u_2^2(n_2) = 1/n_2 - 3/20$	$u_2^3(n_3) = 1/n_3$
Player 3	$u_3^1(n_1) = 2/3n_1$	$u_3^2(n_2) = -10$	$u_3^3(n_3) = 1/n_3$

Let $\beta_1=1, \beta_2=2, \beta_3=3$

	Resource 1	Resource 2	Resource 3
Strategy Profile		2	1 3
Player 1		1 2	3
$u_1^3(4) < u_1^2(3)$			
$0.25 < 0.2722$			
Player 2		1	2 3
$u_2^2(3) < u_2^3(5)$			
$0.1833 < 0.2$			
Player 3	3	1	2
$u_3^3(5) < u_3^1(3)$			
$0.2 < 0.2222$			
Player 1	3		1 2
$u_1^2(1) < u_1^3(3)$			
$0.3166 < 0.3333$			
Player 2	3	2	1
$u_2^3(3) < u_2^2(2)$			
$0.3333 < 0.35$			
Player 3		2	1 3
$u_3^1(3) < u_3^3(4)$			
$0.2222 < 0.25$			

For each player there are effectively only two strategies as the third one is always minimal.

A deviation is considered either as a right to left or left to right move.

	Strategy 1	Strategy 2	Strategy 3
Player 1	$u_1^1(n_1) = -10$	$u_1^2(n_2) = 1/15n_2 + 1/4$	$u_1^3(n_3) = 1/n_3$
Player 2	$u_2^1(n_1) = -10$	$u_2^2(n_2) = 1/n_2 - 3/20$	$u_2^3(n_3) = 1/n_3$
Player 3	$u_3^1(n_1) = 2/3n_1$	$u_3^2(n_2) = -10$	$u_3^3(n_3) = 1/n_3$

Let $\beta_1=1, \beta_2=2, \beta_3=3$

	Resource 1	Resource 2	Resource 3
Strategy Profile		2	1 3
Player 1		1 2	3
$u_1^3(4) < u_1^2(3)$			
$0.25 < 0.2722$			
Player 2		1	2 3
$u_2^2(3) < u_2^3(5)$			
$0.1833 < 0.2$			
Player 3	3	1	2
$u_3^3(5) < u_3^1(3)$			
$0.2 < 0.2222$			
Player 1	3		1 2
$u_1^2(1) < u_1^3(3)$			
$0.3166 < 0.3333$			
Player 2	3	2	1
$u_2^3(3) < u_2^2(2)$			
$0.3333 < 0.35$			
Player 3		2	1 3
$u_3^1(3) < u_3^3(4)$			
$0.2222 < 0.25$			

It is always optimal (unique best reply) for the deviator to play the opposite strategy played by the player preceding him.

As the number of players is odd, a pure strategy Nash equilibrium clearly does not exist.

Sequential version of crowding game

What if the players don't make their choices simultaneously, but rather play one by one?

Given a game G with n-players we consider a *sequential-move* version of G.

We must give an order to the players and since we have $n!$ ways of ordering the players, we have $n!$ different sequential-move versions of G.

Inaccuracy of Nash equilibrium

The Nash equilibrium does not take into account the fact that some of the players are aware of the actions of the players preceding them.

Some Nash equilibria might never arise

Definition *[Subgame Perfect Equilibrium]*: A subgame perfect equilibrium is a strategy profile such that player's strategies constitute a Nash equilibrium in every subgame of the original game.

This notion of equilibrium eliminates Nash equilibria in which the player threats are not credible

Sequentially solvable games

Definition: A (simultaneous-move) game *G* is *sequentially solvable* if each of its sequential-move version has at least one subgame-perfect equilibrium in pure strategies that corresponds also to a Nash equilibrium of G.

Thm[Mil98]: *Every crowding game is sequentially solvable*

Sequentially solvable games

Thus every sequential-move version of a crowding game has a subgame perfect equilibrium in pure strategies such that the state achieved by the equilibrium is an equilibrium of *G*. Such a state is said to be a *sequential move equilibrium* of *G.*

Is every equilibrium of a crowding game also a sequential-move equilibrium? NO

An equilibrium that is not a sequential-move equilibrium

Two player, two strategy crowding game

Payoff: $u_1^1(1)=1, u_1^2(1)=1, u_2^1(1)=1, u_2^2(1)=2, u_i^k(2)=-1$ for every i and k

$$G = \begin{pmatrix} (-1,-1) & (2,2) \\ (1,1) & (-1,-1) \end{pmatrix}$$

The simultaneous-move version has two equilibria (1,2) and (2,1) but only the first one is a sequential move equilibrium.

Sequential move equilibrium

What is sufficient condition for an equilibrium to be a *sequential-move* equilibrium of a crowding game?

Intuitively, in the last example, the reason that the second (social worst) equilibrium is not a sequential move equilibrium is the fact that it is Pareto dominated by the first one.

Definition: A game state *s* <u>*Pareto dominates*</u> another state *s'* if for every *i*, $u_i(s) \geq u_i(s')$ and there is an *i* such that $u_i(s) > u_i(s')$

Strong equilibria

Definition: *An equilibrium is strong if there exist no group of players who can increase their payoffs by simultaneously changing their actions, if the rest of the players do not change their actions.*

Claim[Mil98]: *In every crowding game, an equilibrium which is not Pareto dominated by another equilibrium is strong.*

Strong equilibrium => sequential move equilibrium

Thm[Mil98] *Every strong equilibrium of a crowding game G is a sequential-move equilibrium of G.*

The converse is not true

Weighted vs unweighted crowding game

- For two strategies the FIP hold for unweighted and weighted crowding game

- For two player the FBRP hold for unweighted and weighted crowding game

- In general:
  1. Unweighted crowding games always admit a pure NE
  2. Weighted crowding game need not have even a pure NE

- Every unweighted crowding game is sequentially solvable.

The service provider game

- On the Internet users need increasing bandwidth
- A provider provides bandwidth service to his customers at fixed price
- The customers can accept or reject the offer
- Is there a customers equilibria?
  - Customers game can be viewed as a weighted crowding game
  - Weighted crowding games with only two strategies always admit a pure NE

The service provider game (2)

- What is the best pricing scheme for the provider?
- Which customers will join the service in customers best/worst equilibrium?
- What is the provider's profit in best/worst equilibrium?
- The answers will be provided both with perfect and imperfect information for the provider

SPG: the results for provider’s revenue

- FPTAS for the computation of provider’s revenue under individual price
- If the prices are function of the demands then there is a pseudo-polynomial time algorithm
  - In particular if the prices are equal to the demands the problem is inapproximable
- Average case analysis: FPTAS when demands and “utility threshold” may have arbitrary, continuous probability distribution

Overview

- Congestion and Potential games (Carmine)
- Crowding games (Pasquale)
- Complexity of pure equilibria (Vincenzo)

Finding equilibria

- Fundamental result of Game Theory:
  Every game has a randomized Nash equilibrium
  - The proof is non-constructive (at least not in polytime)
- But how to actually **find** an equilibrium?
  - The complexity of finding a rand. NE is not known precisely
  - Apparently needing exponential time even for two players
- So, what about finding **pure** NE?
  - Which games do always admit pure equilibria?
  - Can we find some pure equilibrium in polynomial time?
  - If the payoff table is given explicitly the problem is trivial

Notation

n: number of players

S_i: actions for player i

$S = S_1 \times \ldots \times S_n$: set of states of the game

$u_i : S \rightarrow R$: utility function of player i

Pure Nash equilibrium (**NE**): state $s \in S$ s.t.

$$u_i(s) \geq u_i(s')$$

for every i and for every state s' differing from s only in the i-th component.

Nash dynamics graph: node set S, edge (s,s') iff s and s' differ only in one component, the i-th, and $u_i(s') > u_i(s)$

If the Nash graph is acyclic, there is at least a pure NE (the sink)

Congestion games

- Set of resources E
- A strategy is a subset of resources: $S_i \subseteq 2^E$
- n_k is the number of players who select resource e_k
- c_k (cost or delay) is a non-decreasing function of n_k
- The cost incurred by a player is the total cost of the resources he selects:

$$u_i(s) = -\sum_{k \in s_i} c_k(n_k(s))$$

- In a **network** congestion game, the resources are edges in a network and S_i is the set of all paths from a source node a_i to a destination node b_i
- In a **symmetric** congestion game, all S_i's are the same

Congestion games and NE

Thm: **Every congestion game has a pure NE.**

Proof:

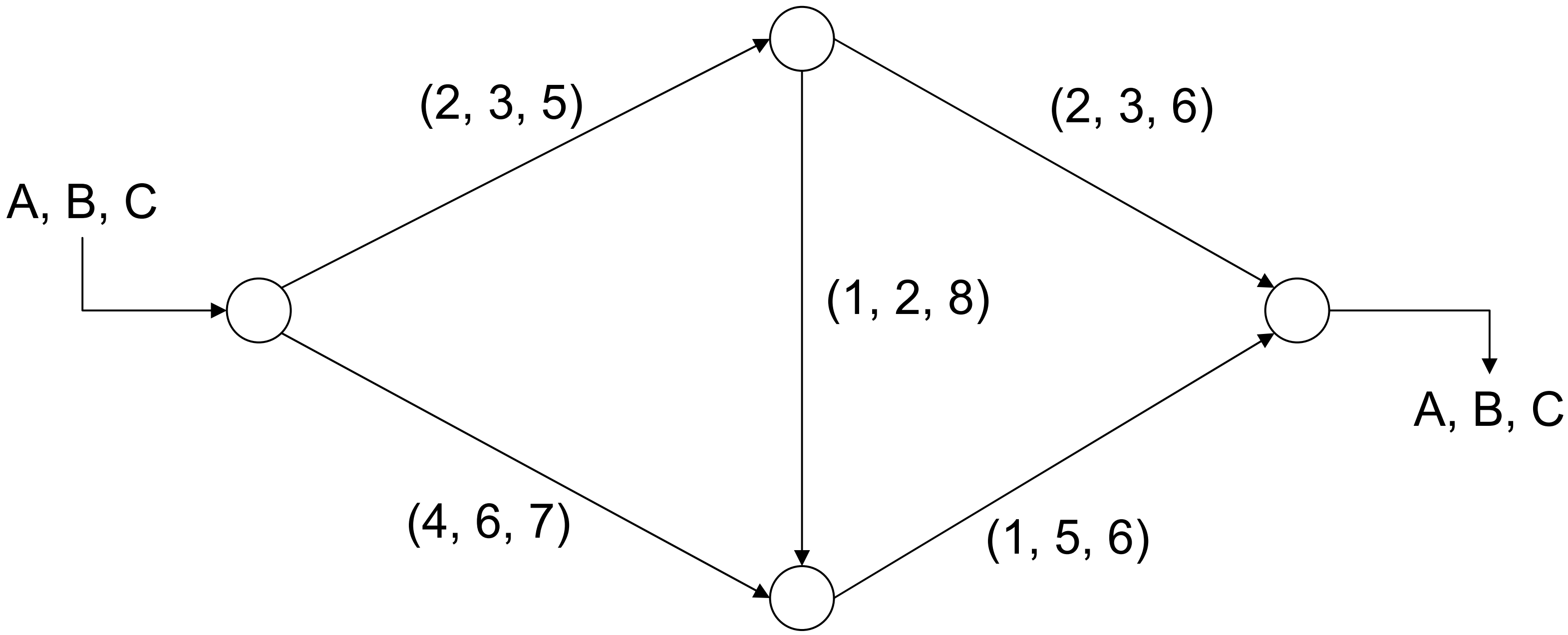
$$\phi(s) = -\sum_k \sum_{j=1}^{n_k(s)} c_k(j)$$

is an **exact potential**, that is, when s and s' differ only in the i-th component,

$$\phi(s) - \phi(s') = u_i(s) - u_i(s')$$

Then any local optimum of ϕ is a pure NE.

An example network congestion game



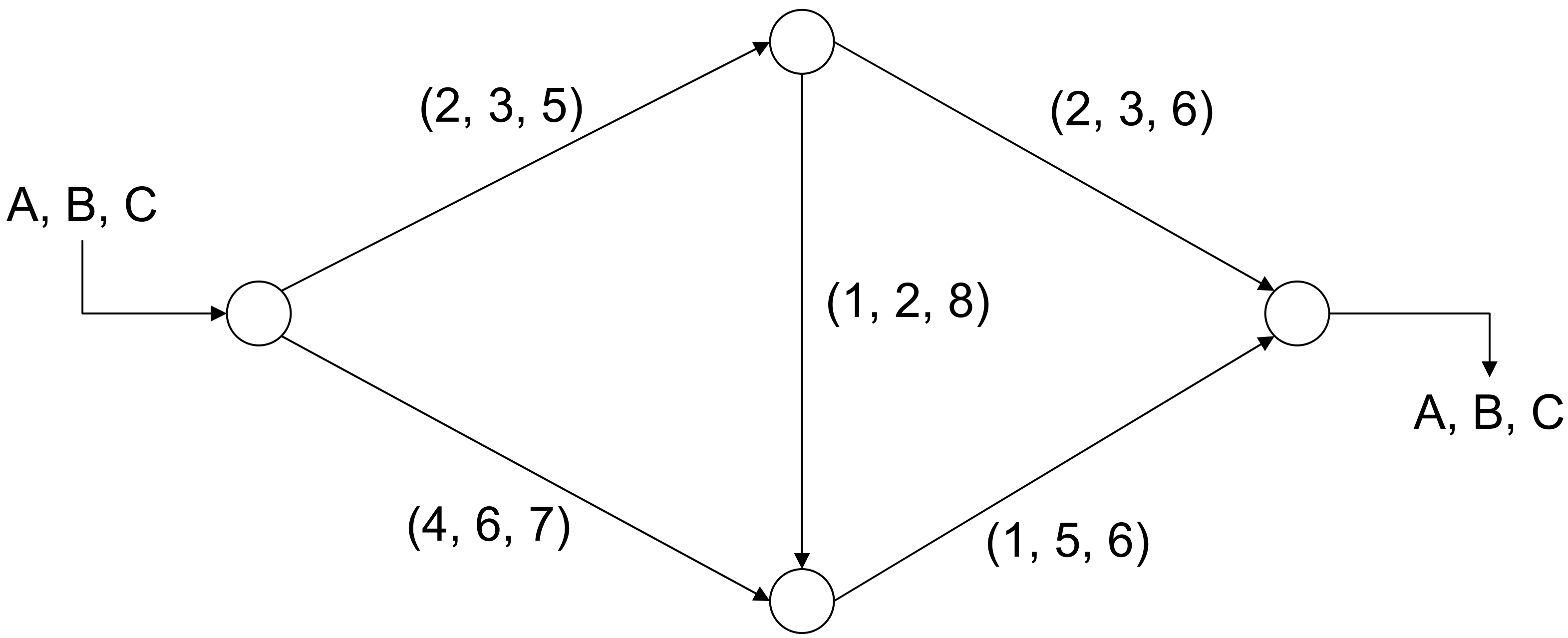
An algorithm for symmetric network congestion games

Thm: There is a poly-time algorithm for computing a pure NE of any symmetric network congestion game.

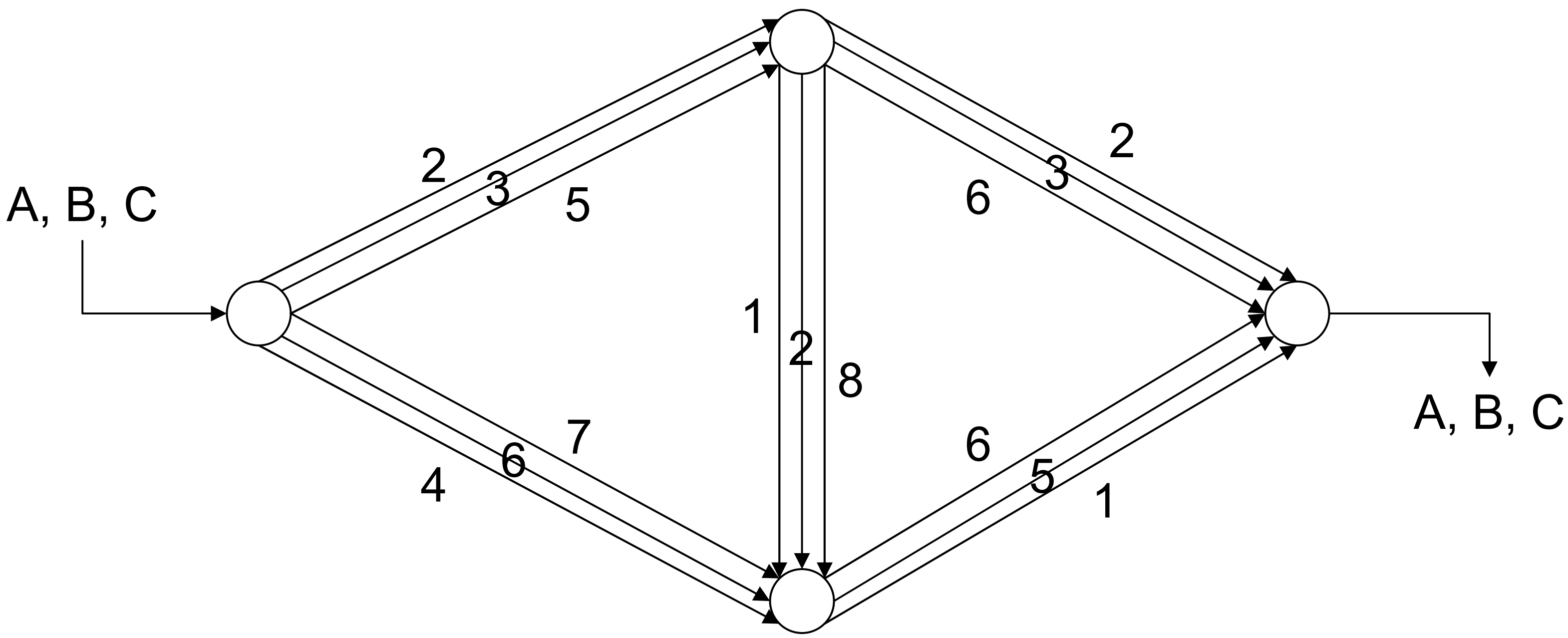
The algorithm computes the optimum of $\phi(s)$.

Algorithm: given a network N=(V,E,a,b) and the delay functions c_e, replace each edge e with n parallel edges between the same nodes, each with capacity 1 and with costs $c_e(1)$, $c_e(2)$, …, $c_e(n)$. Then compute an optimum integer **min-cost flow** of value n.

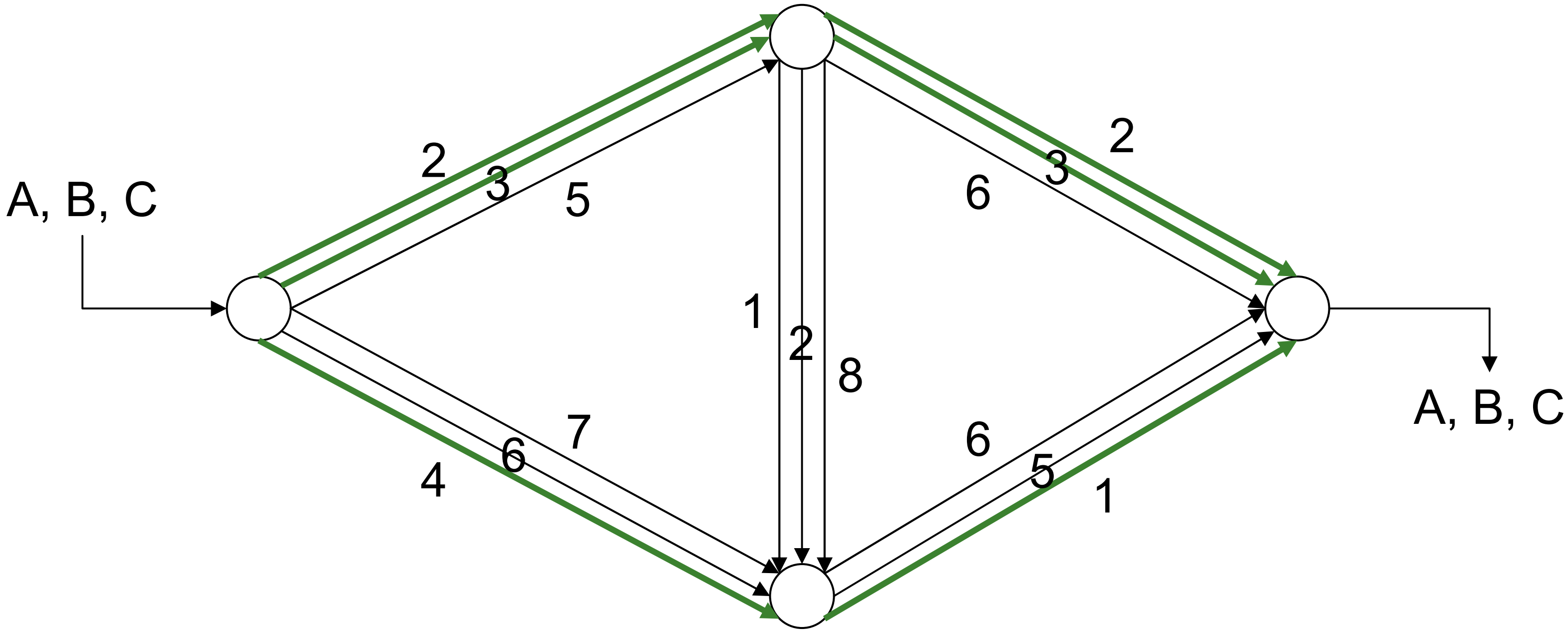
Example



Example



Example



$\min_f \text{cost}(f) = \min_s -\phi(s) = 15$

The non-atomic case

Non-atomic case: network (V,E), k endpoints (a_i,b_i), number of players $\to \infty$;

for each edge e, a non-decreasing delay function $c_e:[0,1] \to R^+$;

k flow requirements r_i, $\sum r_i = 1$;

Lipschitz assumption: $\exists C\ \forall e\ \forall x,y$ s.t. $0 \le x < y \le 1$:

$$|c_e(y)-c_e(x)| \le C|y-x|$$

Flow path: a path with nonzero flow

NE: a k-commodity valid flow s.t. for all pairs a_i-b_i, the delay of any flow path p is no larger than that of any other a_i-b_i path p' :

$$c_p(f) \le c_{p'}(f)$$

ε-approximate NE: $c_p(f) \le c_{p'}(f) + \varepsilon$ for any flow path p and path p'

An algorithm for the non-atomic case

Thm: An ε-approximate NE for a non-atomic congestion game can be computed in time poly(|E|,C,1/ε).

Algorithm: Let $\delta = \varepsilon$ / 4C|E|.

A NE optimizes $\phi(f) = \sum_e \phi_e(f_e)$ where $\phi_e(f_e) = \int_0^{f_e} c_e(t)dt$

Replace each edge e by sequence of parallel arcs of cap. δ and cost

$$\frac{\phi_e(i\delta) - \phi_e((i-1)\delta)}{\delta}$$

per unit flow on the i-th arc, that is, piecewise lin. approximate ϕ_e

Solve a fractional multicommodity min-cost flow on the new graph.

The corresponding flow in the original graph is an ε-approx. NE.

Complexity class PLS

A (minimization) problem in **PLS** is given by:

- a set of instances $I \subseteq \Sigma^*$
- for each $x \in I$, a set of feasible solutions $F_x \subseteq \Sigma^{p(|x|)}$
- a poly-time function $c_x(s)$ (the cost of solution s for instance x); $c_x(s) = \infty$ if s is not in F_x
- for each $x \in I$, $s \in F_x$, a neighborhood $N_x(s) \subseteq F_x$
- a poly-time function $g_x(s)$ such that $g_x(s) = s'$ with $s' \in N_x(s)$ and $c_x(s') < c_x(s)$, or $g_x(s)$="**no**" if no such s' exists

$\Rightarrow$ Finding a pure NE for a cong. game is in PLS

PLS-completeness

A **PLS-reduction** from A to B has also to map every local optimum of B to a local optimum of A.

A sample PLS-complete problem is **POSNAE3FLIP**:

I = sets of weighted 3-clauses with only positive literals

F_x = all truth assignments to variables in x

A clause (a,b,c) is "satisfied" if a,b,c are **not all equal**

$c_x(s)$ = total weight of unsatisfied clauses

$N_x(s)$ = assignments obtained by flipping a variable of s

General congestion games

Thm: **Finding a pure NE in general congestion games is PLS-complete.**

Proof. Let x be an instance of POSNAE3FLIP. Construct a congestion game:

- the players are the variables of x

- to every clause u of weight w associate two resources e_u, e'_u:

- $c_{e_u}(k) = \begin{cases} 0 & \text{if } k \le 2 \\ w & \text{if } k \ge 3. \end{cases}$

- S_i = { {all e_u : u contains i}, {all e'_u : u contains i} }

Any NE of the congestion game is a local optimum of x.

Symmetric congestion games

Thm: **Finding a pure NE in symmetric congestion games is PLS-complete.**

Proof. We reduce the non-symmetric case to the symmetric case. Given a cong. game with action sets $S_1, \ldots, S_n$, let:

- $S'_i = \{ s \cup \{e_i\} : s \in S_i \}$ where e_i's are new resources

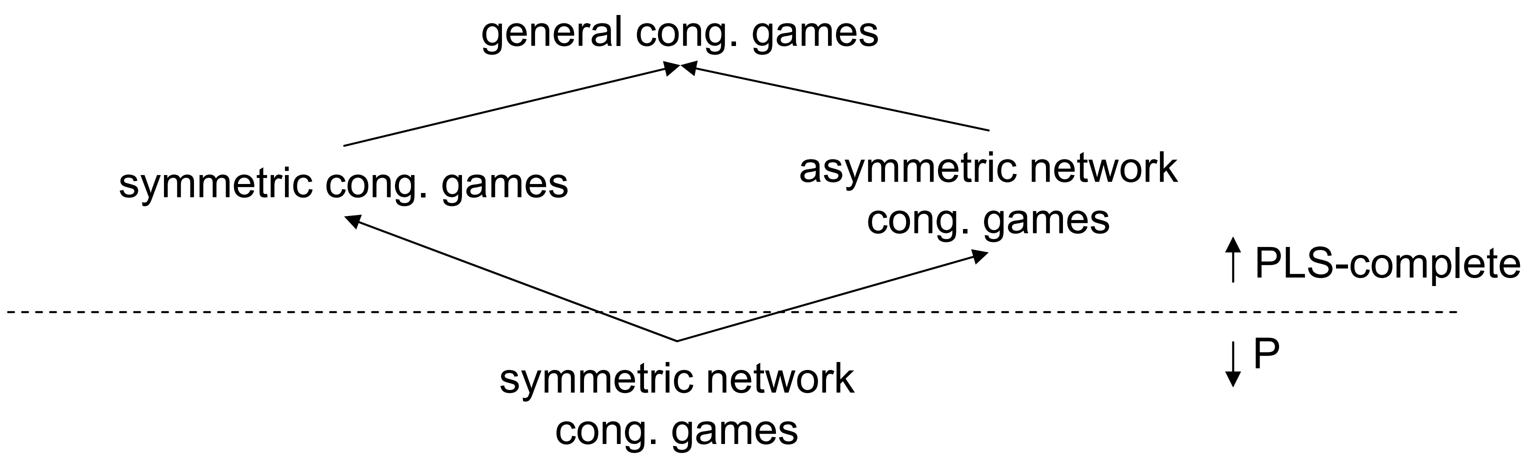
- $c_{e_i}(k) = \begin{cases} 0 & \text{if } k = 1 \\ M & \text{if } k \geq 2. \end{cases}$

Consider a new game with common strategy set $\cup_i S'_i$. The e_i's act as "locks" for the roles of the players. Any NE of the new game will correspond to a NE of the original game.

Asymmetric network congestion games

Thm: **Finding a pure NE in asymmetric network congestion games is PLS-complete.**

Proof. Reworking of "possibly the most complex reduction in the literature after PCP".



PLS and ordinal potential games

Thm: Any exact potential game is isomorphic to a congestion game.

But consider **ordinal potential games**:

$$\mathrm{sgn}(\ \phi(s) - \phi(s')\ ) = \mathrm{sgn}(\ u_i(s) - u_i(s')\ )$$

Finding a pure NE in a family of games with polynomially computable ordinal potentials is in PLS

The converse holds, too: for every problem in PLS there is a family of ordinal potential games having as equilibria exactly the local optima of the problem

Other games with pure equilibria

- Congestion games with subjective delays (crowding games)
- Congestion games with objective player-specific delays:

  n players, m resources

  the cost of resource *e* is a non-decreasing function $c_e(X)$ of the specific **set** X of players choosing *e* (instead of their number |X|)

  Consider the delay vector

  $D = (c_1, c_2, \ldots, c_m)$ where wlog $c_1 \geq c_2 \geq \ldots \geq c_m$

  Defecting player goes from c_a to c_b, $b > a$, but $c'_a > c'_b$

  $\Rightarrow$ the new D is **lexicographically smaller** than the old one

Open problems

- Complexity of finding pure equilibria
  - Congestion games vs. other games
- Potentials
  - The only way to establish convergence of Nash dynamics?
  - Are there convergent games with non-polytime computable potentials?