

# *Frugality in Path Mechanisms*

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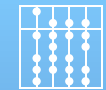
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# *Implementation Theory*

- ▶  $N = \{1, \dots, n\}$  set of **agents** (players)
- ▶  $A = \{a_1, \dots, a_k\}$  set of social **alternatives** (outcomes)
- ▶  $\mathcal{T}$  domain of possible **states** (informations held by agents):
  - $(T, p)$ , where  $T \subseteq \mathcal{T}$  and  $p$  density function on  $T$ , is called **information structure**
  - $T = \{t\}$  means **complete information**
- ▶  $v_i : A \times \mathcal{T} \rightarrow \mathbb{R}$  private **preference** function of agent  $i$   
 $v_i(a, t) > v_i(b, t)$  means agent  $i$  prefers  $a$  over  $b$  in state  $t$
- ▶  $E = (N, A, \{v_i(\cdot, t)\}_{i \in N})$  **environment** (with state  $t$ )

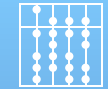


Let  $\mathcal{E}$  be a class of environments (for fixed  $N$  and  $A$ ).

## Social Choice Rules:

- ▶  $F : \mathcal{E} \rightarrow \mathcal{P}(A) \setminus \emptyset$  is an **SCR**
- ▶  $f : \mathcal{E} \rightarrow A$  (total) is an **SCF**

**Write**  $F(t)$  instead of  $F(E)$ , where  $E \in \mathcal{E}$



## Standard:

- ▶  $M_i$  set of **messages** (strategies) for agent  $i \in N$
- ▶  $g : M_1 \times \cdots \times M_n \rightarrow A$  (total) **outcome function**
- ▶  $\Gamma = (\{M_i\}_{i \in N}, g)$  is a **mechanism**

## Using payments:

- ▶  $p_i : M_1 \times \cdots \times M_m \rightarrow \mathbb{R}$  side **payments** to agent  $i$
- ▶  $u_i(a, t) = v_i(a, t) + p_i(m)$  quasi-linear **utility** function of agent  $i$
- ▶  $\Gamma = (\{M_i\}_{i \in N}, g, \{p_i\}_{i \in N})$  is **mechanism** (with side payments)

Let  $\Gamma = (\{M_i\}_{i \in N}, g)$  be a mechanism, and let  $E$  be an environment.

$\mathcal{S}(\Gamma, E) \subseteq M_1 \times \cdots \times M_n$  is **solution concept** for game  $(\Gamma, E)$

Typical solution concepts:

- ▶ **dominant strategy** profiles:  $m_i \in M_i$  is dominant for agent  $i$  iff  $v_i(g(m_i, m_{-i}), t) \geq v_i(g(m'_i, m_{-i}), t)$  for all  $m'_i \in M_i, m_{-i} \in M_{-i}$
- ▶ **Nash equilibria**: strategy profile  $m$  is Nash iff for all  $i \in N$  and for all  $m'_i \in M_i, v_i(g(m_i, m_{-i}), t) \geq v_i(g(m'_i, m_{-i}), t)$
- ▶ **Bayes-Nash equilibria** for incomplete information  
→ Nash is Bayes-Nash with complete information

**Remark:** Direct translation to mechanisms with payments

Let  $\mathcal{E}$  be a class of environments.

- ▶ SCR  $F$  is (fully)  $\mathcal{S}$ -implementable  $\iff_{\text{def}}$  there exists  $\Gamma = (\{M_i\}_{i \in N}, g)$  such that  $g(\mathcal{S}(\Gamma, E)) = F(E)$  for every  $E \in \mathcal{E}$ .
- ▶ SCR  $F$  is weakly  $\mathcal{S}$ -implementable  $\iff_{\text{def}}$  there exists  $\Gamma = (\{M_i\}_{i \in N}, g)$  such that  $g(\mathcal{S}(\Gamma, E)) \subseteq F(E)$  for every  $E \in \mathcal{E}$ .

**For SCF:** Full implementation = weak implementation

- ▶ Direct (revelation) mechanism for SCF  $f$ :

$$M_1 \times \cdots \times M_n = \mathcal{T} \text{ and } f = g.$$

**Theorem [Gibbard-Satterthwaite].** Suppose  $\|A\| \geq 3$  and preference functions admit all strict preference rankings. If an SCF  $f$  is implementable by dominant strategies, then  $f$  is dictatorial.

Directed mechanism is **truthful** (w.r.t. solution concept  $\mathcal{S}$ )  
 $\iff$  for all  $E \in \mathcal{E}$ ,  $v$  is in  $\mathcal{S}$ .

### Revelation principle.

Suppose there is an SCR  $F$  implementable (w.r.t. Nash equilibrium) by a mechanism  $\Gamma = (\{M_i\}_{i \in N}, g, \{p_i\}_{i \in N})$ .  
Then there exists a mechanism  $\Gamma' = (\{M_i\}_{i \in N}, g', \{p'_i\}_{i \in N})$   
such that for all  $E \in \mathcal{E}$ ,

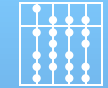
- ▶  $g'(\mathcal{N}(\Gamma', E)) = F(E)$ ,
- ▶  $p_i(m) = p'_i(m)$  for all  $m \in \mathcal{N}(\Gamma', E)$ ,
- ▶ truth-telling  $m = v$  is a strategy in  $\mathcal{N}(\Gamma', E)$ .

# ***Path Mechanisms***

Let  $G = (V, E)$  be any bi-connected (multi)graph,  $s, t \in V$ . We have one packet to send over  $G$  from  $s$  to  $t$ .

- ▶  $N = E$  (i.e., agents are edges)
- ▶  $A = \{(r_0, r_1, \dots, r_\ell) \mid \ell \geq 1, r_0 = s, r_\ell = t, \{r_j, r_{j-1}\} \in E\}$   
(i.e., set of all  $s$ - $t$  paths)
- ▶  $\mathcal{T} = \{(c_1, \dots, c_n) \mid c_j \in \mathbb{R}_+\}$  (i.e.,  $c_i$  is edge  $i$ 's transit cost per packet)
- ▶ edge  $i$ 's preference function is

$$v_i(a, t) = \begin{cases} -c_i & \text{if edge } i \text{ belongs to path } a \in A, \\ 0 & \text{otherwise} \end{cases}$$



Social Choice Rule:

- ▶  $F(t) = \arg \max_{a \in A} \sum_{i \in N} v_i(a, t)$

VCG implementation:

- ▶  $M_i = \mathbb{R}_+$  for all  $i$  (direct revelation mechanism)
- ▶ agents report messages  $(m_1, \dots, m_n)$
- ▶ outcome  $g(m) =$  any shortest path in  $G$  with edge weights  $m_j$
- ▶ payment  $p_i(m) = d_{G|m_i=\infty}(s, t) - d_{G|m_i=0}(s, t)$

**Remark:** VCG is weak implementation

# Shortest-Path Implementation (II)

**Theorem.** VCG truthfully implements  $F$ .

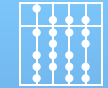
Proof: Consider agent  $i$  and messages  $(m_i, m_{-i})$  with  $m_i = v_i$ .

- ▶ Suppose  $i \in g(m_i, m_{-i})$ . Then

$$\begin{aligned} u_i &= m_i + d_{G|m_i=\infty}(s, t) - d_{G|m_i=0}(s, t) \\ &= d_{G|m_i=\infty}(s, t) - d_G(s, t) \geq 0. \end{aligned}$$

Consider message  $m'_i \neq v_i$ .

- If  $i \in g(m'_i, m_{-i})$  then  $u_i(g(m'_i, m_{-i})) = u_i(g(m_i, m_{-i}))$ .
  - If  $i \notin g(m'_i, m_{-i})$  then  $u_i(g(m'_i, m_{-i})) = 0$ .
- ▶ Case  $i \notin g(m_i, m_{-i})$  using similar arguments.



Overall payments for VCG:

$$C = \sum_{i \in N} p_i(m) = \sum_{i \in g(m)} d_{G|m_i=\infty}(s, t) - d_{G|m_i=0}(s, t)$$

How large can  $C$  be?

**Example 1.** Let  $G$  consist of two parallel edges  $P$  and  $Q$  between  $s$  and  $t$ ,  $c(P) \leq c(Q)$ .

$\implies$  VCG chooses  $P$  (with payment equal to second-best alternative).

Thus,

$$C = c(Q) = c(P) + (c(Q) - c(P)).$$

# Analyzing Total VCG Payments (II)

**Example 2.** Let  $G$  consist of two (node-)disjoint paths  $P$  and  $Q$  of length  $k$  between  $s$  and  $t$ ,  $c(P) \leq c(Q)$ .

$\implies$  VCG chooses  $P$

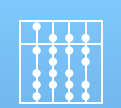
Total payment:

$$\begin{aligned} C &= k \cdot c(Q) - (k - 1) \cdot c(P) \\ &= c(P) + k \cdot (c(Q) - c(P)) \end{aligned}$$

**Interpretation.**

Even if the alternatives are very close in costs (tight market), the total payment is not:

For fixed  $c(P) = L$  and  $c(Q) = L(1 + \varepsilon)$ , we obtain  $C = O(k)$ .



# ***Frugality***

## Frugal Path Problem: [Archer, Tardos, 2002]

Are there (truthful) path mechanisms with significantly lower worst-case payments than for VCG?

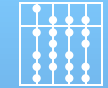
### Fundamental property of truthful mechanisms:

Depending on the message vector  $m$  and outcome  $g$ , there exists a threshold bid

$$m_i^*(g, m_{-i}) = \inf\{m_i \mid i \in g(m_i, m_{-i})\}$$

$\implies$  For truthfulness, we pay this threshold bid, if  $i \in g(m)$ .

**Consequence.** We can only vary over path selection rule  $g$  (this means loss in social welfare)



Theorem. [Elkind, Sahai, Steiglitz, 2004]

Any truthful path mechanism  $\Gamma$  (w.r.t. dominant strategies) induces total payments of at least

$$c(P) + \frac{1}{2} \cdot k \cdot |c(Q) - c(P)|$$

in the worst case, where

- ▶  $c(P)$  is the cost of the best path w.r.t. to  $\Gamma$
- ▶  $c(Q)$  is the cost of the second-best path w.r.t. to  $\Gamma$
- ▶  $k$  is the number of edges in  $P$

# Dominant Strategy Implementation (II)

## Proof idea:

Consider graph  $G$  having two (node-)disjoint paths  $P$  and  $Q$  of length  $k$  ( $k$  even).

Define  $\varepsilon =_{\text{def}} |c(Q) - c(P)|$ .

Let  $m^{P,i}$  denote the message vector of path  $P$  edges, defined as follows

$$m_j^{P,i} =_{\text{def}} \begin{cases} -\frac{c(P)}{k} & \text{if } i \neq j \\ -\frac{c(P)}{k} - \varepsilon & \text{if } i = j \end{cases}$$

Message vector  $m^{Q,i}$  is defined in the same way.

# Dominant Strategy Implementation (III)

Fix any truthful path mechanism  $\Gamma$ .

Consider the directed (complete) bipartite graph  $G_\Gamma$  with:

- ▶ vertex sets  $\{m^{P,1}, \dots, m^{P,k}\}$  and  $\{m^{Q,1}, \dots, m^{Q,k}\}$
- ▶ edge set:

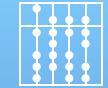
$$(m^{P,i}, m^{Q,j}) \in E(G_\Gamma) \iff Q = g(m^{P,i}, m^{Q,j})$$

$$(m^{Q,i}, m^{P,j}) \in E(G_\Gamma) \iff P = g(m^{P,i}, m^{Q,j})$$

We have  $\|V(G_\Gamma)\| = 2k$  and  $\|E(G_\Gamma)\| = k^2$ .

$\implies$  there exists a vertex with out-degree at least  $\ell = \frac{k}{2}$ ,

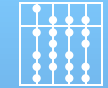
W.l.o.g. let  $m^{Q,1}$  be such vertex, let  $m^{P,i_1}, \dots, m^{P,i_t}$  be the endpoints of the out-going edges, i.e.,  $P = g(m^{P,i_j}, m^{Q,1})$ .



What are the payments to edges in  $P$ ?

- ▶ each edge in  $P$  is paid at least its bid  $\frac{c(P)}{k}$
- ▶ each edge  $i_j$  with  $(m^{Q,1}, m^{P,i_j}) \in E(G_\Gamma)$  is paid its threshold bid, i.e.,  $\frac{c(P)}{k} + \varepsilon$ .

$$\implies C \geq c(P) + \frac{k}{2} \cdot \varepsilon = c(P) + \frac{k}{2} \cdot |c(Q) - c(P)|.$$



## Remarks:

- ▶ Lower bound holds for all bi-connected graphs (by embedding of the bad example)
- ▶ There are examples with total payments almost  $c(P) + k \cdot (c(Q) - c(P))$  for any truthful mechanism
- ▶ Lower bound holds for randomized mechanisms as well

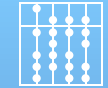
# Bayes-Nash Equilibrium Implementation (I)

Scenario with incomplete information.

- ▶  $T_i = [0, \omega_i]$  is the set of possible transit costs for edge  $i$
- ▶  $f_i$  density function on  $T_i$
- ▶  $\mathcal{T} = T_1 \times \cdots \times T_n$  set of possible states
- ▶  $f = \prod_{i \in N} f_i$  density function on  $\mathcal{T}$

**Note:** We omit prior beliefs and conditional probabilities

# Bayes-Nash Equilibrium Implementation (II)



Probability that edge  $i$  will be in the winning path:

$$q_i(m_i) = \int_{M_{-i}} Q_i(m_i, m_{-i}) \cdot f_{-i}(m_{-i}) dm_{-i}$$

Expected payment to edge  $i$ :

$$p_i(m_i) = \int_{M_{-i}} P_i(m_i, m_{-i}) \cdot f_{-i}(m_{-i}) dm_{-i}$$

Expected utility of edge  $i$  with private cost  $-c_i = m_i$ :

$$U_i(c_i) = -c_i \cdot q_i(c_i) + p_i(c_i).$$

# Bayes-Nash Equilibrium Implementation (III)

Mechanism is **optimal** iff  $\sum_{i \in N} \mathbf{E}_{T_i} p_i(c_i)$  is minimal.

- ▶ **Regular** mechanism design problem:  $c_i + \frac{F_i(c_i)}{f_i(c_i)}$  is nondecreasing for all  $i \in N$

(Background: Mechanism is truthful iff all  $q_i$ 's are nonincreasing.)

- ▶  $x_i(c_i) = c_i + \frac{F_i(c_i)}{f_i(c_i)}$  is called **virtual cost** of edge  $i$

# Bayes-Nash Equilibrium Implementation (IV)

Theorem. [Elkind, Sahai, Steiglitz, 2004], [Myerson, 1981]

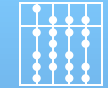
For regular mechanism design problems, the optimal mechanism  $(\{M_i\}_{i \in N}, Q, P)$  is given by an allocation rule  $Q$  and a payment rule  $P$  such that for all message vectors  $m$ ,

- ▶  $Q(m)$  is the path with smallest virtual costs

- ▶ 
$$P_i(m) = Q_i(m) \cdot m_i + \int_{-m_i}^{\omega_i} Q_i(r, m_{-i}) dr.$$

**Remark:** Payments correspond to threshold bids of an agent

# Bayes-Nash Equilibrium Implementation (V)



In some cases payments in optimal mechanism are significantly lower than payments in VCG!

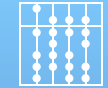
**Example.** Exponential distribution  $f(x) = e^{-x}$ :

- ▶ VCG overpayment per edge  $\Omega(\sqrt{n})$
- ▶ Optimal mechanism overpayment per edge  $O(\log n)$

# ***Summary***

What do we have learnt?

- ▶ Truthfulness in dominant strategies can cause very high cost (even in tight markets)
- ▶ There is no significantly cheaper mechanism in dominant strategies than VCG (even if social welfare is allowed to be suboptimal)
- ▶ Optimal mechanisms for Bayes-Nash equilibria can do cheaper in many cases than VCG does



Some open questions:

- ▶ What about frugality for different solution concepts (without guaranteeing truthfulness)?
- ▶ What about frugality for repeated games?
- ▶ What about frugality for all-pairs shortest path problems? Formulation as combinatorial auctions?
- ▶ What about frugality for policy routing (not all packet are allowed to go over each edge, as e.g., in BGP)?