Frugality in Path Mechanisms

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Implementation Theory
Environments

- $N = \{1, \ldots, n\}$ set of agents (players)
- $A = \{a_1, \ldots, a_k\}$ set of social alternatives (outcomes)
- $T$ domain of possible states (informations held by agents):
  - $(T, p)$, where $T \subseteq T$ and $p$ density function on $T$, is called information structure
  - $T = \{t\}$ means complete information
- $v_i : A \times T \to \mathbb{R}$ private preference function of agent $i$
  - $v_i(a, t) > v_i(b, t)$ means agent $i$ prefers $a$ over $b$ in state $t$
- $E = (N, A, \{v_i(\cdot, t)\}_{i \in N})$ environment (with state $t$)
Let $\mathcal{E}$ be a class of environments (for fixed $N$ and $A$).

Social Choice Rules:

- $F : \mathcal{E} \rightarrow \mathcal{P}(A) \setminus \emptyset$ is an SCR
- $f : \mathcal{E} \rightarrow A$ (total) is an SCF

Write $F(t)$ instead of $F(E)$, where $E \in \mathcal{E}$
Mechanisms

Standard:

- $M_i$ set of messages (strategies) for agent $i \in N$
- $g: M_1 \times \cdots \times M_n \rightarrow A$ (total) outcome function
- $\Gamma = (\{M_i\}_{i \in N}, g)$ is a mechanism

Using payments:

- $p_i : M_1 \times \cdots \times M_m \rightarrow \mathbb{R}$ side payments to agent $i$
- $u_i(a, t) = v_i(a, t) + p_i(m)$ quasi-linear utility function of agent $i$
- $\Gamma = (\{M_i\}_{i \in N}, g, \{p_i\}_{i \in N})$ is mechanism (with side payments)
Let \( \Gamma = (\{M_i\}_{i \in N}, g) \) be a mechanism, and let \( E \) be an environment.

\[ S(\Gamma, E) \subseteq M_1 \times \cdots \times M_n \] is solution concept for game \((\Gamma, E)\).

Typical solution concepts:

- **dominant strategy profiles**: \( m_i \in M_i \) is dominant for agent \( i \) iff 
  \[ v_i(g(m_i, m_{-i}), t) \geq v_i(g(m'_i, m_{-i}), t) \]
  for all \( m'_i \in M_i, m_{-i} \in M_{-i} \).

- **Nash equilibria**: strategy profile \( m \) is Nash iff for all \( i \in N \) and for all \( m'_i \in M_i, v_i(g(m_i, m_{-i}), t) \geq v_i(g(m'_i, m_{-i}), t) \).

- **Bayes-Nash equilibria** for incomplete information
  \[ \longrightarrow \text{Nash is Bayes-Nash with complete information} \]

**Remark**: Direct translation to mechanisms with payments.
Implementability

Let $\mathcal{E}$ be a class of environments.

- **SCR $F$ is (fully) $S$-implementable** $\iff$ there exists $\Gamma = (\{M_i\}_{i \in N}, g)$ such that $g(S(\Gamma, E)) = F(E)$ for every $E \in \mathcal{E}$.

- **SCR $F$ is weakly $S$-implementable** $\iff$ there exists $\Gamma = (\{M_i\}_{i \in N}, g)$ such that $g(S(\Gamma, E)) \subseteq F(E)$ for every $E \in \mathcal{E}$.

For SCF: Full implementation $=$ weak implementation

- Direct (revelation) mechanism for SCF $f$:
  $$M_1 \times \cdots \times M_n = \mathcal{T} \text{ and } f = g.$$

**Theorem [Gibbard-Satterthwaite].** Suppose $\|A\| \geq 3$ and preference functions admit all strict preference rankings. If an SCF $f$ is implementable by dominant strategies, then $f$ is dictatorial.
Directed mechanism is **truthful** (w.r.t. solution concept $S$) \[ \iff \text{for all } E \in \mathcal{E}, \text{ } v \text{ is in } S. \]

**Revelation principle.**

Suppose there is an SCR $F$ implementable (w.r.t. Nash equilibriums) by a mechanism $\Gamma = (\{M_i\}_{i \in N}, g, \{p_i\}_{i \in N})$. Then there exists a mechanism $\Gamma' = (\{M_i\}_{i \in N}, g', \{p'_i\}_{i \in N})$ such that for all $E \in \mathcal{E},$

\begin{itemize}
  \item $g'(N(\Gamma', E)) = F(E),$
  \item $p_i(m) = p'_i(m)$ for all $m \in N(\Gamma', E),$ \\
  \item truth-telling $m = v$ is a strategy in $N(\Gamma', E).$
\end{itemize}
Path Mechanisms
Let $G = (V, E)$ be any bi-connected (multi)graph, $s, t \in V$. We have one packet to send over $G$ from $s$ to $t$.

- $N = E$ (i.e., agents are edges)
- $A = \{(r_0, r_1, \ldots, r_\ell) \mid \ell \geq 1, r_0 = s, r_\ell = t, \{r_j, r_{j-1}\} \in E\}$ (i.e., set of all $s$-$t$ paths)
- $T = \{(c_1, \ldots, c_n) \mid c_j \in \mathbb{R}_+\}$ (i.e., $c_i$ is edge $i$’s transit cost per packet)
- edge $i$’s preference function is
  \[ v_i(a, t) = \begin{cases} 
  -c_i & \text{if edge } i \text{ belongs to path } a \in A, \\
  0 & \text{otherwise}
  \end{cases} \]
Shortest-Path Implementation (I)

Social Choice Rule:

\[ F(t) = \arg \max_{a \in A} \sum_{i \in N} v_i(a, t) \]

VCG implementation:

\[ M_i = \mathbb{R}_+ \text{ for all } i \] (direct revelation mechanism)

agents report messages \((m_1, \ldots, m_n)\)

outcome \(g(m) = \) any shortest path in \(G\) with edge weights \(m_j\)

payment \(p_i(m) = d_{G|m_i=\infty}(s, t) - d_{G|m_i=0}(s, t)\)

Remark: VCG is weak implementation
Shortest-Path Implementation (II)

**Theorem.** VCG truthfully implements $F$.

**Proof:** Consider agent $i$ and messages $(m_i, m_{-i})$ with $m_i = v_i$.

- Suppose $i \in g(m_i, m_{-i})$. Then

  $$u_i = m_i + d_{G|m_i=\infty}(s, t) - d_{G|m_i=0}(s, t)$$
  $$= d_{G|m_i=\infty}(s, t) - d_{G}(s, t) \geq 0.$$ 

  Consider message $m'_i \neq v_i$.

  - If $i \in g(m'_i, m_{-i})$ then $u_i(g(m'_i, m_{-i})) = u_i(g(m_i, m_{-i}))$.
  - If $i \notin g(m'_i, m_{-i})$ then $u_i(g(m'_i, m_{-i})) = 0$.

- Case $i \notin g(m_i, m_{-i})$ using similar arguments.
Analyzing Total VCG Payments (I)

Overall payments for VCG:

\[ C = \sum_{i \in N} p_i(m) = \sum_{i \in g(m)} d_{G|m_i=\infty}(s, t) - d_{G|m_i=0}(s, t) \]

How large can \( C \) be?

Example 1. Let \( G \) consist of two parallel edges \( P \) and \( Q \) between \( s \) and \( t \), \( c(P) \leq c(Q) \).

\( \implies \) VCG chooses \( P \) (with payment equal to second-best alternative).

Thus,

\[ C = c(Q) = c(P) + (c(Q) - c(P)) \]
Example 2. Let $G$ consist of two (node-)disjoint paths $P$ and $Q$ of length $k$ between $s$ and $t$, $c(P) \leq c(Q)$.

$\implies$ VCG chooses $P$

Total payment:

$$C = k \cdot c(Q) - (k - 1) \cdot c(P)$$

$$= c(P) + k \cdot (c(Q) - c(P))$$

Interpretation.
Even if the alternatives are very close in costs (tight market), the total payment is not:

For fixed $c(P) = L$ and $c(Q) = L(1 + \varepsilon)$, we obtain $C = O(k)$. 
Frugality
Frugality

Frugal Path Problem: [Archer, Tardos, 2002]

Are there (truthful) path mechanisms with significantly lower worst-case payments than for VCG?

Fundamental property of truthful mechanisms:

Depending on the message vector $m$ and outcome $g$, there exists a threshold bid

$$m_i^*(g, m_{-i}) = \inf \{m_i \mid i \in g(m_i, m_{-i})\}$$

$\implies$ For truthfulness, we pay this threshold bid, if $i \in g(m)$.

Consequence. We can only vary over path selection rule $g$ (this means loss in social welfare)
Theorem. [Elkind, Sahai, Steiglitz, 2004]

Any truthful path mechanism $\Gamma$ (w.r.t. dominant strategies) induces total payments of at least

$$c(P) + \frac{1}{2} \cdot k \cdot |c(Q) - c(P)|$$

in the worst case, where

- $c(P)$ is the cost of the best path w.r.t. to $\Gamma$
- $c(Q)$ is the cost of the second-best path w.r.t. to $\Gamma$
- $k$ is the number of edges in $P$
Dominant Strategy Implementation (II)

Proof idea:
Consider graph $G$ having two (node-)disjoint paths $P$ and $Q$ of length $k$ ($k$ even).
Define $\varepsilon = |c(Q) - c(P)|$.
Let $m^{P,i}$ denote the message vector of path $P$ edges, defined as follows

$$
 m^{P,i}_j = \begin{cases} 
 -\frac{c(P)}{k} & \text{if } i \neq j \\
 -\frac{c(P)}{k} - \varepsilon & \text{if } i = j 
\end{cases}
$$

Message vector $m^{Q,i}$ is defined in the same way.
Fix any truthful path mechanism $\Gamma$.

Consider the directed (complete) bipartite graph $G_\Gamma$ with:

- vertex sets $\{m^{P,1}, \ldots, m^{P,k}\}$ and $\{m^{Q,1}, \ldots, m^{Q,k}\}$
- edge set:

$$(m^{P,i}, m^{Q,j}) \in E(G_\Gamma) \iff Q = g(m^{P,i}, m^{Q,j})$$
$$(m^{Q,i}, m^{P,j}) \in E(G_\Gamma) \iff P = g(m^{P,i}, m^{Q,j})$$

We have $\|V(G_\Gamma)\| = 2k$ and $\|E(G_\Gamma)\| = k^2$.

$\implies$ there exists a vertex with out-degree at least $\ell = \frac{k}{2}$.

W.l.o.g. let $m^{Q,1}$ be such vertex, let $m^{P,i_1}, \ldots, m^{P,i_t}$ be the endpoints of the out-going edges, i.e., $P = g(m^{P,i_j}, m^{Q,1})$. 
What are the payments to edges in $P$?

- each edge in $P$ is paid at least its bid $\frac{c(P)}{k}$
- each edge $i_j$ with $(m^{Q,1}, m^{P,i_j}) \in E(G_T)$ is paid its threshold bid, i.e., $\frac{c(P)}{k} + \varepsilon$.

$$\implies C \geq c(P) + \frac{k}{2} \cdot \varepsilon = c(P) + \frac{k}{2} \cdot |c(Q) - c(P)|.$$
Dominant Strategy Implementation (V)

Remarks:

- Lower bound holds for all bi-connected graphs (by embedding of the bad example)
- There are examples with total payments almost $c(P) + k \cdot (c(Q) - c(P))$ for any truthful mechanism
- Lower bound holds for randomized mechanisms as well
Bayes-Nash Equilibrium Implementation (I)

Scenario with incomplete information.

- $T_i = [0, \omega_i]$ is the set of possible transit costs for edge $i$
- $f_i$ density function on $T_i$
- $\mathcal{T} = T_1 \times \cdots \times T_n$ set of possible states
- $f = \prod_{i\in N} f_i$ density function on $\mathcal{T}$

*Note:* We omit prior beliefs and conditional probabilities
Bayes-Nash Equilibrium Implementation (II)

Probability that edge \( i \) will be in the winning path:

\[
q_i(m_i) = \int_{M_{-i}} Q_i(m_i, m_{-i}) \cdot f_{-i}(m_{-i}) \, dm_{-i}
\]

Expected payment to edge \( i \):

\[
p_i(m_i) = \int_{M_{-i}} P_i(m_i, m_{-i}) \cdot f_{-i}(m_{-i}) \, dm_{-i}
\]

Expected utility of edge \( i \) with private cost \(-c_i = m_i\):

\[
U_i(c_i) = -c_i \cdot q_i(c_i) + p_i(c_i).
\]
Mechanism is **optimal** iff \( \sum_{i \in N} E_{T_i} p_i(c_i) \) is minimal.

- **Regular** mechanism design problem: \( c_i + \frac{F_i(c_i)}{f_i(c_i)} \) is nondecreasing for all \( i \in N \)
  
  (Background: Mechanism is truthful iff all \( q_i \)'s are nonincreasing.)

- \( x_i(c_i) = c_i + \frac{F_i(c_i)}{f_i(c_i)} \) is called **virtual cost** of edge \( i \)
Bayes-Nash Equilibrium Implementation (IV)

Theorem. [Elkind, Sahai, Steiglitz, 2004], [Myerson, 1981]

For regular mechanism design problems, the optimal mechanism \((\{M_i\}_{i \in N}, Q, P)\) is given by an allocation rule \(Q\) and a payment rule \(P\) such that for all message vectors \(m\),

- \(Q(m)\) is the path with smallest virtual costs

\[
P_i(m) = Q_i(m) \cdot m_i + \int_{-m_i}^{\omega_i} Q_i(r, m_{-i}) \, dr.
\]

Remark: Payments correspond to threshold bids of an agent
In some cases payments in optimal mechanism are significantly lower than payments in VCG!

**Example.** Exponential distribution $f(x) = e^{-x}$:

- VCG overpayment per edge $\Omega(\sqrt{n})$
- Optimal mechanism overpayment per edge $O(\log n)$
Summary
What do we have learnt?

- Truthfulness in dominant strategies can cause very high cost (even in tight markets)
- There is no significantly cheaper mechanism in dominant strategies than VCG (even if social welfare is allowed to be suboptimal)
- Optimal mechanisms for Bayes-Nash equilibria can do cheaper in many cases than VCG does
Questions

Some open questions:

- What about frugality for different solution concepts (without guaranteeing truthfulness)?
- What about frugality for repeated games?
- What about frugality for all-pairs shortest path problems? Formulation as combinatorial auctions?
- What about frugality for policy routing (not all packet are allowed to go over each edge, as e.g., in BGP)?