The most interesting results known about the KP-model

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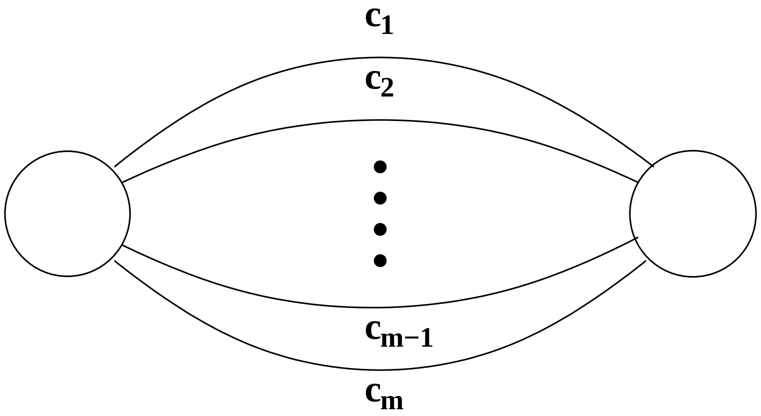
Content

The KP-model: Instance

We consider a model introduced by Koutsoupias and Papadimitriou [KP99]:

- n users with traffic of size w_i for all $i \in [n]$. We denote by $\mathbf{w} = (w_1, \ldots, w_n)$ the traffic vector. If $w_{i_1} = w_{i_2}$ for all $i_1, i_2 \in [n], i_1 \neq i_2$, the users are identical.
- 2 nodes, source and sink, connected by m related links with capacity c_j for all $j \in [m]$. If $c_{j_1} = c_{j_2}$ for all $j_1, j_2 \in [m], j_1 \neq j_2$, the links are identical. We denote by $\mathbf{c} = (c_1, \ldots, c_m)$ the capacity vector.



Each user $i \in [n]$

- wants to route his traffic w_i from source to sink via one of the m edges,
- assigns (as his strategy) his traffic to link $j \in [m]$ with probability p_{ij}, and
- tries to minimize his individual cost, that is, his routing time.

The KP-model: Strategies

In a pure strategy profile, denoted $\mathbf{L} = \langle \ell_1, \ell_2, \ldots, \ell_n \rangle$, each user $i \in [n]$ chooses a specific link ℓ_i. In a mixed strategy profile, denoted $\mathbf{P} = (p_{ij})$, each user $i \in [n]$ chooses link $j \in [m]$ with probability p_{ij}. We denote

$$\text{support}(i) \quad = \quad \{j \in [m] \mid p_{ij} > 0\}$$

In a fully mixed strategy profile, denoted $\mathbf{F} = (f_{ij})$, $f_{ij} > 0$ for all $i \in [n]$, $j \in [m]$.

The KP-model: Individual Cost

Let $\mathbf{P}$ be an arbitrary mixed strategy profile.
The routing time of user i on link j, called expected latency cost for user i on link j, is

$$\lambda_{ij} = \frac{w_i + \sum_{k \neq i} p_{kj} w_k}{c_j}.$$

The minimum expected latency cost for user i is

$$\lambda_i = \min_{j \in [m]} \lambda_{ij}.$$

The KP-model: Load and Latency

The expected load on link j is defined by

$$\delta_j = \sum_{i \in [n]} p_{ij} w_i .$$

For each link $j \in [m]$, define the expected latency Λ_j as the expected traffic on link j. Thus,

$$\Lambda_j = \sum_{i \in [n]} \frac{p_{ij} w_i}{c_j} = \frac{\delta_j}{c_j} .$$

The maximum expected latency is $\Lambda = \max_{j \in [m]} \Lambda_j$.

The KP-Model: Social Cost

For a mixed strategy profile $\mathbf{P}$ the social cost, denoted $\mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{P})$, is the expected maximum latency on a link, where the expectation is taken over all random choices of the users. Thus,

$$\mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) = \sum_{\langle \ell_1, \ell_2, \ldots, \ell_n \rangle \in [m]^n} \left(\prod_{k=1}^{n} p_{k\ell_k} \cdot \max_{j \in [m]} \frac{\sum_{k:\ell_k = j} w_k}{c_j} \right).$$

The social optimum, denoted $\mathsf{OPT}(\mathbf{w}, \mathbf{c})$, is the least possible value, over all pure strategy profiles $\mathbf{L}$, of the social cost. Thus,

$$\mathsf{OPT}(\mathbf{w}, \mathbf{c}) = \min_{\mathbf{L}} \mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{L}).$$

The KP-model: Nash Equilibria

We call user i satisfied if $\lambda_{ij} = \lambda_i$ for all $j \in [m]$ with $p_{ij} > 0$, otherwise unsatisfied. So a satisfied user only chooses links with minimum expected latency cost.

$\mathbf{P}$ is a Nash equilibrium if no user $i \in [n]$ is unsatisfied, that is,

$$\lambda_{ij} = \lambda_i \quad \forall j \in [m] \text{ with } p_{ij} > 0.$$

Note: The definition of Nash equilibria is independent of the definition of social cost!!!

The KP-model: Price of Anarchy

A best (worst) Nash equilibrium is a Nash equilibrium $\mathbf{P}$ that minimizes (maximizes) SC$(\mathbf{w}, \mathbf{c}, \mathbf{P})$. The worst social cost is the social cost of a worst Nash equilibrium and is denoted by WC$(\mathbf{w}, \mathbf{c})$.

The price of anarchy, also called coordination ratio, is the maximum value, over all traffic vectors w, of the ratio

$$\frac{\mathsf{WC}(\mathbf{w}, \mathbf{c})}{\mathsf{OPT}(\mathbf{w}, \mathbf{c})}.$$

The KP-model: The Fully Mixed Nash Equilibrium Conjecture

Conjecture 1. [GLMMS03] *Consider the model of arbitrary users and related links. Then, for any problem instance* $(\mathbf{w}, \mathbf{c})$ *such that the fully mixed Nash equilibrium* $\mathbf{F}$ *exists, and for any Nash equilibrium* $\mathbf{P}$*,*

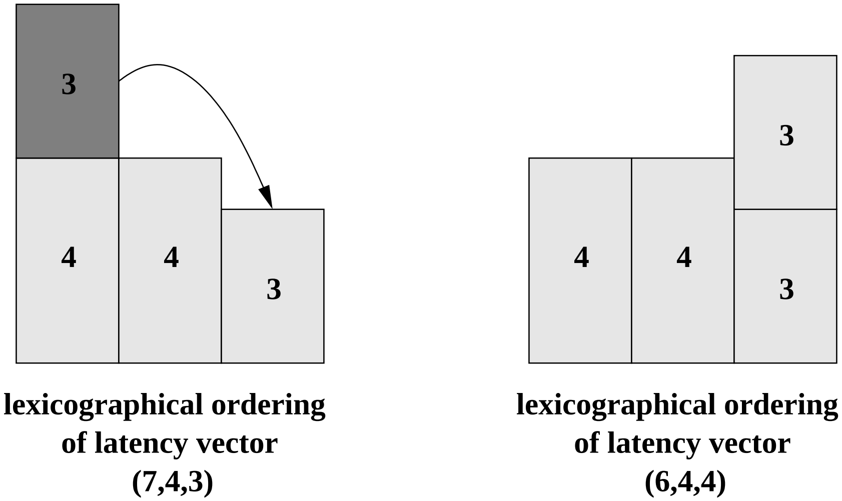
$$\mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{P}) \leq \mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{F}).$$

In the following, we denote this conjecture as FMNE Conjecture.

Selfish Steps (1)

In a selfish step, exactly one unsatisfied user is allowed to change his pure strategy. A selfish step is a greedy selfish step if the user chooses his best strategy.

It is easy to see that the lexicographical ordering of the latency vector decreases in each selfish step. Thus, starting with any pure strategy profile, every sequence of (greedy) selfish steps eventually ends in a pure Nash equilibrium.



Theorem 1. [FKKMS02] *Consider the model of arbitrary users and related links. Then, for any problem instance* $(\mathbf{w}, \mathbf{c})$*, there exists at least one pure Nash equilibrium.*

Selfish Steps (2)

Selfish steps can be used to compute a pure Nash equilibrium from any given pure strategy profile without altering the social cost. We call such an approach nashification [FGLMR03a].

Theorem 2. [FKKMS02] *Consider the model of arbitrary users and related links. Then, for any problem instance* $(\mathbf{w}, \mathbf{c})$*, there exists at least one pure Nash equilibrium* $\mathbf{L}$ *with* $\mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{L}) = \mathsf{OPT}(\mathbf{w}, \mathbf{c})$.

Identical Links: Computation of Pure Nash Equilibria (1)

In order to directly compute a pure Nash equilibrium for a given instance, the LPT-algorithm, introduced by Graham [Gra69] can be used.

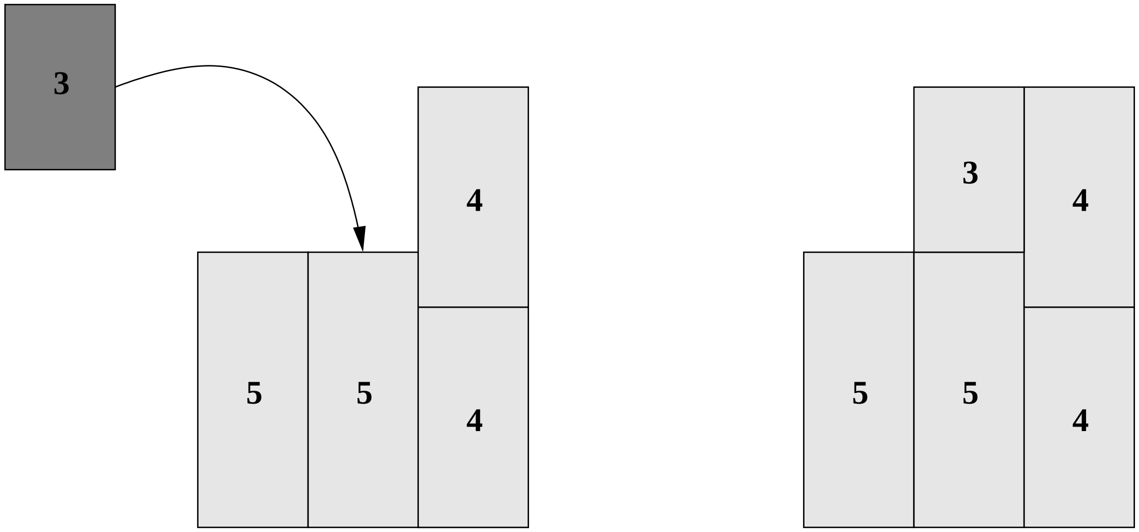
Theorem 3. [FKKMS02] *Consider the model of arbitrary users and identical links. Then, for any problem instance* $(\mathbf{w}, m)$, LPT *computes a pure Nash equilibrium* $\mathbf{L}$ *with*

$$\mathsf{SC}(\mathbf{w}, m, \mathbf{L}) \leq \left(\frac{4}{3} - \frac{1}{3m}\right) \cdot \mathsf{OPT}(\mathbf{w}, m),$$

using $O((n+m)\log(m))$ *time.*

Identical Links: Computation of Pure Nash Equilibria (2)

Sketch of Proof: Prove by induction on $i \in [n]$ that after assigning the first i users according to LPT, the system is in a Nash equilibrium.



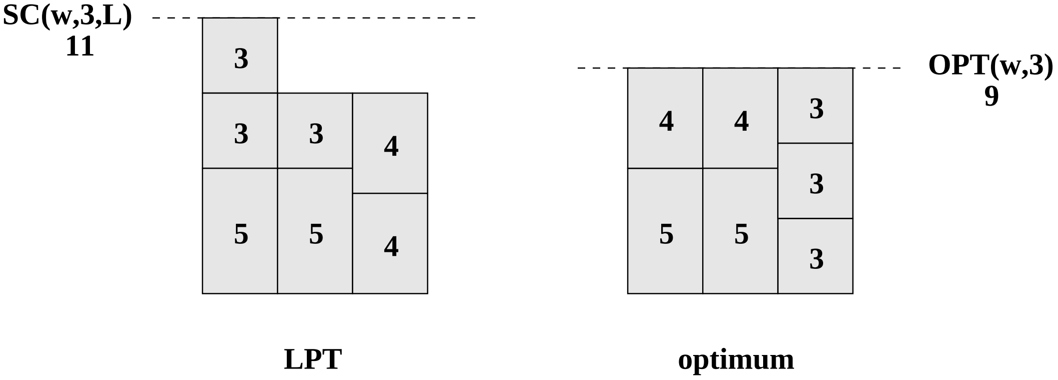
Identical Links: Computation of Pure Nash Equilibria (3)

Example 1. *Consider the following instance:*

- $n = 7$ *users,* $m = 3$ *identical links*
- $w_1 = w_2 = 5$,
  $w_3 = w_4 = 4$, *and*
  $w_5 = w_6 = w_7 = 3$.

LPT *returns a pure Nash equilibrium* $\mathbf{L}$ *with social cost* 11 *whereas the optimum pure Nash equilibrium has social cost* 9. *Thus,*

$$\frac{\mathsf{SC}(\mathbf{w}, m, \mathbf{L})}{\mathsf{OPT}(\mathbf{w}, m)} = \frac{11}{9} = \frac{4}{3} - \frac{1}{3m}.$$



Identical Links: Computation of Pure Nash Equilibria (4)

As seen, every sequence of greedy selfish steps converges to a Nash equilibrium. However, there exist sequences of greedy selfish steps with exponential length.

Theorem 4. [EKM03] *Consider the model of arbitrary users and identical links. Then, there exists an instance* $(\mathbf{w}, m)$ *for which the maximum length of a sequence of greedy selfish steps is at least*

$$\frac{\left(\frac{n}{m-1}\right)^{m-1}}{2(m-1)!}.$$

Identical Links: Computation of Pure Nash Equilibria (5)

The following upper bound on the number of greedy selfish steps can be proved.

Theorem 5. [FGLMR03a] *Consider the model of arbitrary users and identical links. Then, for any problem instance* $(\mathbf{w}, m)$*, the length of a sequence of greedy selfish steps is at most* $2^n - 1$ *before reaching a Nash equilibrium.*

Sketch of Proof: Prove by induction on $i \in [n]$ that user i can make at most 2^{i-1} greedy selfish steps:

- User 1 can make at most 1 greedy selfish step.
- User i can only become unsatisfied by greedy selfish steps of users with larger traffic:

$$\sum_{1 \leq k \leq i-1} 2^{k-1} + 1 = 2^{i-1}$$

Identical Links: Computation of Pure Nash Equilibria (6)

Consider the following decision problem:

NASHIFY

INSTANCE:	A problem instance $(\mathbf{w}, m)$, a pure strategy profile $\mathbf{L}$ for the system of the users, and a positive integer k.
QUESTION:	Is there a sequence of at most k selfish steps that transforms $\mathbf{L}$ to a pure Nash equilibrium?

If k is a constant the problem is denoted as k-NASHIFY.

Theorem 6. [GLMMS03] *Consider the model of arbitrary users and identical links. Then,* NASHIFY *is $\mathcal{NP}$-complete even if $m = 2$.*

Sketch of Proof: By reduction from PARTITION.

Identical Links: Computation of Pure Nash Equilibria (7)

Though there exist sequences of greedy selfish steps of exponential length, and though NASHIFY is $\mathcal{NP}$-complete, it is possible to use greedy selfish steps to compute a Nash equilibrium in polynomial time:

NASHIFY-IDENTICAL
Input: a problem instance $(\mathbf{w}, m)$, a pure strategy profile $\mathbf{L}$
Output: a pure strategy profile $\mathbf{L}'$ that is a Nash equilibrium

```
begin
sort the user traffics in non-increasing order so that $w_1 \geq \ldots \geq w_n$;
for each user $i := 1$ to $n$, do
    remove user $i$ from the link he is currently assigned;
    find the link $j$ with the minimum load;
    reassign user $i$ to the link $j$;
od
return the resulting pure strategy profile $\mathbf{L}'$;
end
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Identical Links: Computation of Pure Nash Equilibria (8)

In order to prove the correctness of $\textsc{Nashify-Identical}$, we show:

Lemma 1. [EKM03,GLMMS03] *Consider the model of arbitrary users and identical links. Then, a greedy selfish step of an unsatisfied user* i_1 *with traffic* w_{i_1} *makes no user* i_2 *with traffic* $w_{i_2} \geq w_{i_1}$ *unsatisfied.*

Proof: Let $\mathbf{L} = \langle \ell_1, \ldots, \ell_n \rangle$ be a pure strategy profile, let $j_1 = \ell_{i_1}$, and let j_2 be the link to which user i_1 moves. Assume that user i_2 becomes unsatisfied due to the move of user i_1. Since only the load on link j_1 and j_2 changed, we have to distinguish between two cases:

Identical Links: Computation of Pure Nash Equilibria (9)

1. First, assume that $\ell_{i_2} \neq j_2$, and that user i_2 wants to change his strategy to j_1. Since user i_1 changed his strategy from j_1 to j_2 we know that $\frac{\delta_{j_1}}{c_{j_1}} > \frac{\delta_{j_2}+w_{i_1}}{c_{j_2}}$. Therefore,

   $$\frac{\delta_{l_{i_2}}}{c_{l_{i_2}}} > \frac{\delta_{j_1} - w_{i_1} + w_{i_2}}{c_{j_1}} > \frac{\delta_{j_2} + w_{i_1}}{c_{j_2}} + \frac{w_{i_2} - w_{i_1}}{c_{j_1}} \geq \frac{\delta_{j_2} + w_{i_2}}{c_{j_2}}.$$

   So if user i_2 wants to change the strategy to j_1, the user i_2 was already unsatisfied before user i_1 changed his strategy, a contradiction.
2. For the case that the strategy of user i_2 is j_2, we have for all $j \in [m] \setminus \{j_2\}$ that

   $$\frac{\delta_j + w_{i_2}}{c_j} \geq \frac{\delta_j + w_{i_1}}{c_j} \geq \frac{\delta_{j_2} + w_{i_1}}{c_{j_2}} = \frac{(\delta_{j_2} + w_{i_1} - w_{i_2}) + w_{i_2}}{c_{j_2}}.$$

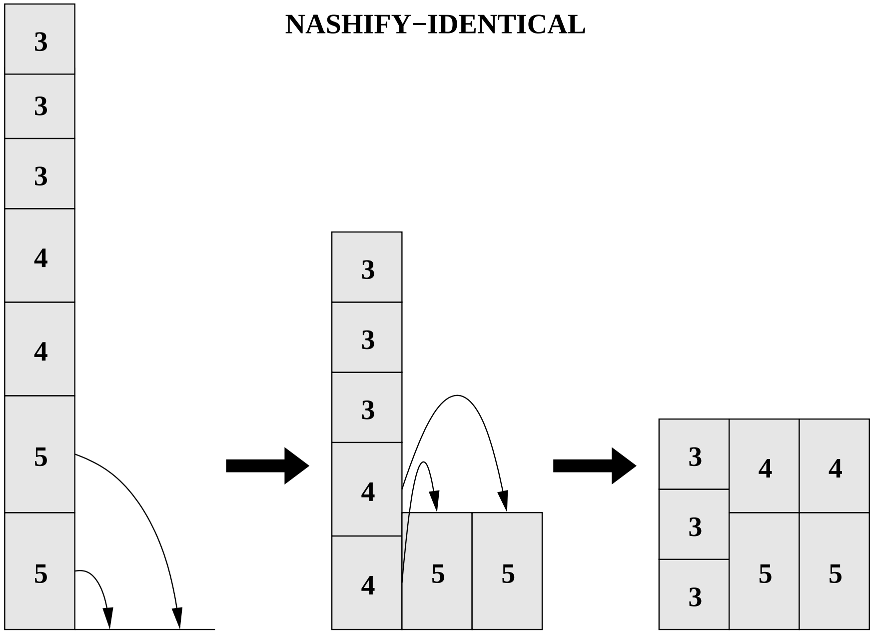
   Therefore, i_2 stays satisfied.

Identical Links: Computation of Pure Nash Equilibria (10)

Theorem 7. [EKM03,GLMMS03] *Consider the model of arbitrary users and identical links. Then, for any problem instance* $(\mathbf{w}, m)$ *and pure strategy profile* $\mathbf{L}$ *with social cost* $\mathsf{SC}(\mathbf{w}, m, \mathbf{L})$*, algorithm* $\textsc{Nashify-Identical}$ *computes a Nash equilibrium from* $\mathbf{L}$ *with social cost at most* $\mathsf{SC}(\mathbf{w}, m, \mathbf{L})$ *using at most* n *greedy selfish steps and* $O(n \log n)$ *time.*

Identical Links: Computation of Pure Nash Equilibria (11)

Example 1 (continued) *For the given instance and the pure strategy profile where all users are assigned to the first link,* Nashify-Identical *uses* 4 *greedy selfish steps before reaching a pure Nash equilibrium.*



Identical Links: Computation of Best Pure Nash Equilibria (1)

One might ask for a polynomial time algorithm to compute a pure Nash equilibrium with minimum social cost.

BEST PURE NASH EQUILIBRIUM

INSTANCE: A problem instance $(\mathbf{w}, \mathbf{c})$, and a positive integer B.

QUESTION: Is there a pure Nash equilibrium $\mathbf{L}$ with SC$(\mathbf{w}, m, \mathbf{L}) < B$?

Theorem 8. [FKKMS02] *Consider the model of arbitrary users and identical links. Then,* BEST PURE NASH EQUILIBRIUM *is $\mathcal{NP}$-complete.*

Sketch of Proof: By reduction from BIN PACKING.

Identical Links: Computation of Best Pure Nash Equilibria (2)

However, the algorithm Nashify-Identical enables us to use any approximation algorithm for scheduling n *jobs* on m identical *machines*. In particular, we can give a PTAS for BEST PURE NASH EQUILIBRIUM by proceeding as follows:

1. Run the PTAS of Hochbaum and Shmoys [HS87]. This yields a pure strategy profile $\mathbf{L}$ such that
$$\mathsf{SC}(\mathbf{w}, m, \mathbf{L}) \leq (1+\varepsilon) \cdot \mathsf{OPT}(\mathbf{w}, m).$$
2. Apply the algorithm Nashify-Identical on $\mathbf{L}$. This yields a Nash equilibrium $\mathbf{L}'$ such that
$$\mathsf{SC}(\mathbf{w}, m, \mathbf{L}') \leq \mathsf{SC}(\mathbf{w}, m, \mathbf{L}') \leq (1+\varepsilon) \cdot \mathsf{OPT}(\mathbf{w}, m).$$

Theorem 9. [GLMMS03] *Consider the model of arbitrary users and identical links. Then, there exists a* PTAS *for* BEST PURE NASH EQUILIBRIUM.

Identical Links: Price of Anarchy and Computation of Worst Pure Nash Equilibria (1)

One might ask for a polynomial time algorithm to compute a pure Nash equilibrium with maximum social cost.

WORST PURE NASH EQUILIBRIUM

INSTANCE:	A problem instance $(\mathbf{w}, \mathbf{c})$, and a positive integer B.
MEASURE:	Is there a pure Nash equilibrium $\mathbf{L}$ with SC$(\mathbf{w}, \mathbf{c}, \mathbf{L}) > B$?

If m is constant the problem is denoted as m-WORST PURE NASH EQUILIBRIUM.

Theorem 10. [FKKMS02] *Consider the model arbitrary users and identical links. Then,* WORST PURE NASH EQUILIBRIUM *is $\mathcal{NP}$-complete*.

Sketch of Proof: By reduction from PARTITION.

Identical Links: Price of Anarchy and Computation of Worst Pure Nash Equilibria (2)

We start with proving the tight upper bound on the price of anarchy.

Theorem 11. [GLMMS03] *Consider the model of arbitrary users and identical links, restricted to pure Nash equilibria. Then, the price of anarchy is*

$$\max_{\mathbf{w},\mathbf{L}} \frac{\mathsf{SC}(\mathbf{w}, m, \mathbf{L})}{\mathsf{OPT}(\mathbf{w}, m)} = 2 - \frac{2}{m+1}.$$

Identical Links: Price of Anarchy and Computation of Worst Pure Nash Equilibria (3)

Proof: Upper bound: Let $\mathbf{L}$ be a pure Nash equilibrium with social cost $\mathsf{SC}(\mathbf{w}, m, \mathbf{L}) = \delta_j$. If there is only one user on link j, then $\mathsf{SC}(\mathbf{w}, m, \mathbf{L}) = \mathsf{OPT}(\mathbf{w}, m)$. So assume that there are at least two users on the link j with maximum load δ_j. Denote w_{min} the minimal traffic of a user on j. Since we consider a Nash equilibrium, the load on the other $m - 1$ links is at least $\delta_j - w_{min}$. So,

$$W \geq \delta_j + (m-1)(\delta_j - w_{min}),$$

and we get

$$\mathsf{OPT}(\mathbf{w}, m) \geq \frac{W}{m} \geq \frac{\delta_j + (m-1)(\delta_j - w_{min})}{m}.$$

But this implies

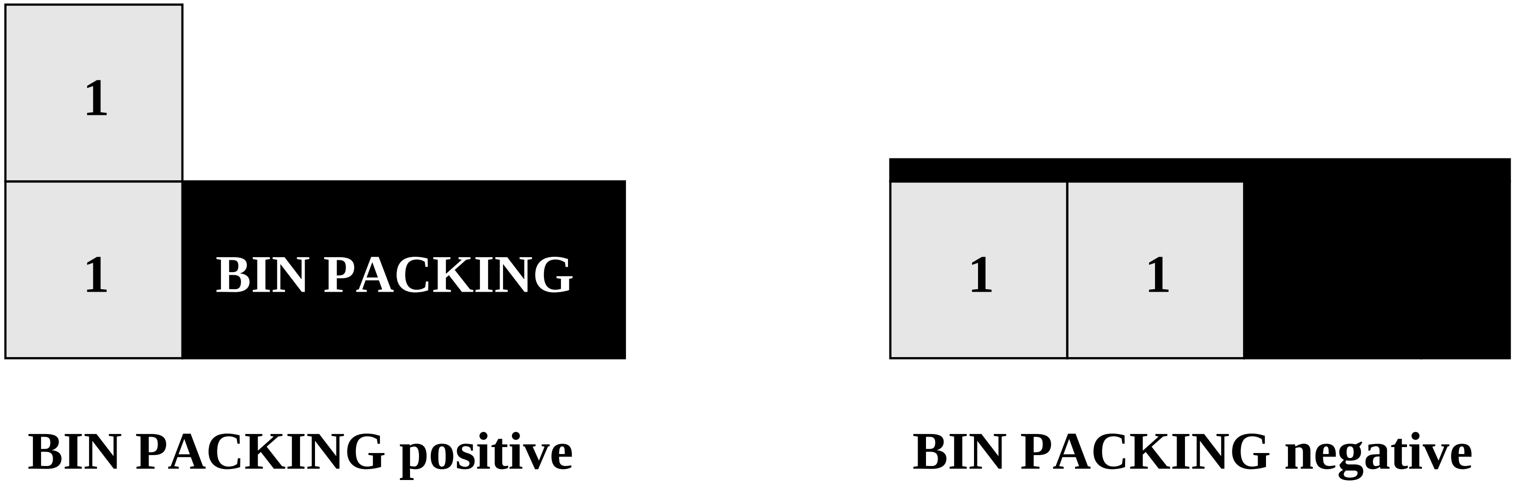
$$\frac{\mathsf{SC}(\mathbf{w}, m, \mathbf{L})}{\mathsf{OPT}(\mathbf{w}, m)} \leq \frac{m\delta_j}{m\delta_j - (m-1)w_{min}}.$$

Since this expression is strictly increasing in w_{min} and because $w_{min} \leq \frac{\delta_j}{2}$, the bound holds.

Identical Links: Price of Anarchy and Computation of Worst Pure Nash Equilibria (4)

Theorem 12. [GLMMS03] *Consider the model of arbitrary users and identical links. If, for any* $\varepsilon > 0$, WORST PURE NASH EQUILIBRIUM *is* $(2 - \frac{2}{m+1} - \varepsilon)$*-approximable, then* $\mathcal{P} = \mathcal{NP}$.

Proof: By reduction from BIN PACKING.



Identical Links: Mixed Nash Equilibria (1)

In contrast to pure Nash equilibria, it is hard even to compute the social cost for a mixed Nash equilibrium However, Fotakis *et al*. [FKKMS02] show that there exists a fully polynomial, randomized approximation scheme for NASH EQUILIBRIUM SOCIAL COST.

Theorem 13. [FKKMS02] *Consider the model of arbitrary users and identical links. Then,* NASH EQUILIBRIUM SOCIAL COST *is* $\#\mathcal{P}$*-complete*.

Theorem 14. [FKKMS02] *Consider the model of arbitrary users and identical links. Then, there exists a fully polynomial, randomized approximation scheme for* NASH EQUILIBRIUM SOCIAL COST.

Identical Links: Mixed Nash Equilibria (2)

Theorem 15. [KP99] *Consider the model of identical users and identical links. Then, the price of anarchy is bounded by*

$$\max_{\mathbf{w},\mathbf{P}} \frac{\mathsf{SC}(\mathbf{w}, m, \mathbf{P})}{\mathsf{OPT}(\mathbf{w}, m)} = \Omega\left(\frac{\log m}{\log(\log(m))}\right).$$

Sketch of Proof: The fully mixed Nash equilibrium can be interpreted as throwing m Balls into m Bins. A well-known result is that the expected maximum number of balls in the bins is $O\left(\frac{\log(m)}{\log(\log(m))}\right)$.

Identical Links: Mixed Nash Equilibria (3)

Theorem 16. [CV02,KMS02] *Consider the model of arbitrary users and identical links. Then, the price of anarchy is bounded by*

$$\max_{\mathbf{w},\mathbf{P}} \frac{\mathsf{SC}(\mathbf{w}, m, \mathbf{P})}{\mathsf{OPT}(\mathbf{w}, m)} \leq \Gamma^{-1}(m) + \Theta(1) = O\left(\frac{\log(m)}{\log(\log(m))}\right).$$

This upper bound is tight up to an additive constant.

Identical Links: Fully Mixed Nash Equilibria

Lemma 2. [MS01] *Consider the model of arbitrary users and identical links. Then, there is a unique fully mixed Nash equilibrium* $\mathbf{P}$ *with* $p_{ij} = \frac{1}{m}$ *for any user* $i \in [n]$ *and link* $j \in [m]$.

Sketch of Proof: The Nash conditions of a fully mixed Nash equilibrium lead to a system of inequalities linear in the probabilities p_{ij}. In case of identical links, one can show that $p_{ij} = \frac{1}{m}$ for all $i \in [n]$ and $j \in [m]$ is a unique solution.

The following theorems provide evidence for the FMNE Conjecture.

Theorem 17. [GLMMS03] *Consider the model of arbitrary users and identical links. Then, for any pure Nash equilibrium* $\mathbf{L}$*, it is* $\mathsf{SC}(\mathbf{w}, m, \mathbf{L}) \leq \mathsf{SC}(\mathbf{w}, m, \mathbf{F})$.

Theorem 18. [LMMRSV03] *Consider the model of identical users and identical links, and assume that* $m = 2$ *and* n *is even. Then, the* FMNE *Conjecture is valid.*

Related Links: Computation of Pure Nash Equilibria (1)

Fotakis *et al.* [FKKMS02] show that the LPT-algorithm returns a pure Nash equilibrium. Friesen [Fri87] proved that its price of anarchy lies between 1.52 and $\frac{5}{3}$.

Theorem 19. [FKKMS02] *Consider the model of arbitrary users on related links. Then, for any problem instance* $(\mathbf{w}, m)$, LPT *computes a pure Nash equilibrium* $\mathbf{L}$ *with*

$$\mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{L}) \leq \frac{5}{3} \cdot \mathsf{OPT}(\mathbf{w}, \mathbf{c}),$$

using $O((n+m)\log(m))$ *time.*

Related Links: Computation of Pure Nash Equilibria (2)

Lemma 3. [FGLMR03a] *Consider the model of arbitrary users and related links. Then, a greedy selfish step of an unsatisfied user* i_1 *with traffic* w_{i_1} *from a link* j_1 *to a link* j_2 *with* $c_{j_1} \leq c_{j_2}$ *makes no user* i_2 *with traffic* $w_{i_2} \geq w_{i_1}$ *unsatisfied.*

Theorem 20. [FGLMR03a] *Consider the model of arbitrary users and related links. Let* $\mathbf{L}$ *be a pure strategy profile with social cost* $\mathsf{SC}(\mathbf{w}, \mathbf{L})$*. Then, there exists an algorithm* NASHIFY-RELATED *that computes a Nash equilibrium from* $\mathbf{L}$ *with social cost at most* $\mathsf{SC}(\mathbf{w}, \mathbf{L})$*, using at most* $(m+1)n$ *moves and* $\mathcal{O}(m^2 n)$ *time.*

Combining the PTAS of Hochbaum and Shmoys [HS88] and NASHIFY-RELATED yields a method for approximating the best Nash equilibrium.

Theorem 21. [FGLMR03a] *Consider the model of arbitrary users and related links. Then, there exists a* PTAS *for* BEST PURE NASH EQUILIBRIUM.

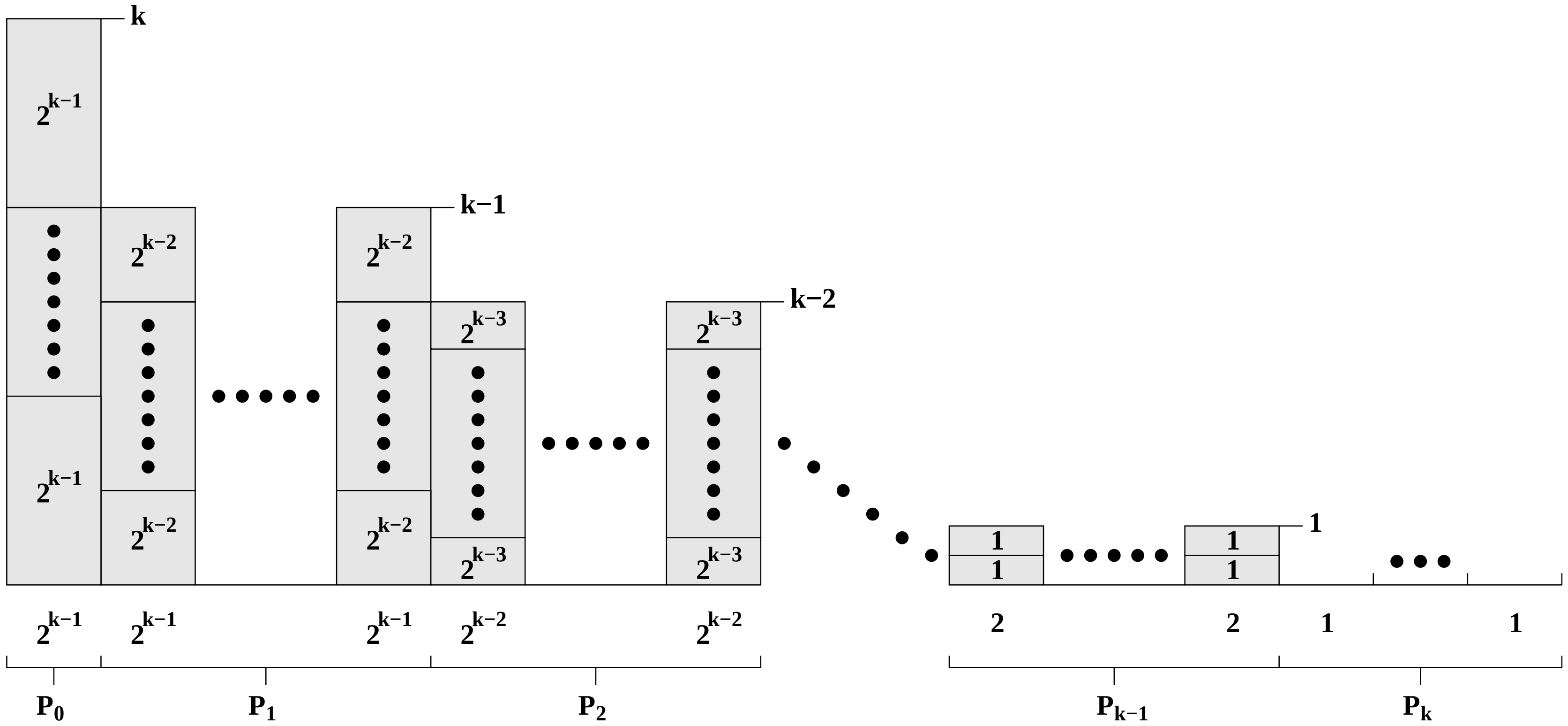
Related Links: Price of Anarchy and Computation of Worst Pure Nash Equilibria (1)

Czumaj and Vöcking [CV02] show the following bound on the price of anarchy which is tight up to an additive constant.

Theorem 22. [CV02] *Consider the model of arbitrary users and related links, restricted to pure Nash equilibria. Then, the price of anarchy is*

$$\begin{aligned} \max_{\mathbf{w},\mathbf{c},\mathbf{L}} \frac{\mathsf{SC}(\mathbf{w},\mathbf{c},\mathbf{L})}{\mathsf{OPT}(\mathbf{w},\mathbf{c})} &\leq \min\left\{\Gamma^{-1}(m)+1\ ,\ 2\cdot\log\left(\frac{c_1}{c_m}\right)+O(1)\right\} \\ &= O\left(\min\left\{\frac{\log(m)}{\log(\log(m))}\ ,\ \log\left(\frac{c_1}{c_m}\right)\right\}\right). \end{aligned}$$

Related Links: Price of Anarchy and Computation of Worst Pure Nash Equilibria (2)



Related Links: Mixed Nash Equilibria

Theorem 23. [CV02] *Consider the model of arbitrary users and related links. Then, the price of anarchy is*

$$\max_{\mathbf{w},\mathbf{c},\mathbf{P}} \frac{\mathsf{SC}(\mathbf{w},\mathbf{c},\mathbf{P})}{\mathsf{OPT}(\mathbf{w},\mathbf{c})} = \Theta\left(\min\left\{\frac{\log m}{\log\log\log m}, \frac{\log m}{\log\left(\frac{\log m}{\log c_1/c_m}\right)}\right\}\right).$$

Sketch of Proof:

- Find a bound on the maximum expected latency on a link.
- Apply the Hoeffding inequality to derive from this bound a bound on the expected maximum latency.

Related Links: Fully Mixed Nash Equilibria (1)

In contrast to the model of identical links, there does not necessarily exist a fully mixed Nash equilibrium, but if it exists, it is unique and can be efficiently computed.

Theorem 24. [MS01] *Consider the model of arbitrary users and related links. Then, there exists a fully mixed Nash equilibrium* $\mathbf{F}$ *if and only if*

$$\left(1 - \frac{mc_j}{C}\right) \cdot \left(1 - \frac{W}{(n-1)w_i}\right) + \frac{c_j}{C} \in (0,1) \quad \forall i \in [n], j \in [m].$$

If $\mathbf{F}$ *exists,* $\mathbf{F}$ *is unique and has associated Nash probabilities*

$$p_{ij} = \left(1 - \frac{mc_j}{C}\right) \cdot \left(1 - \frac{W}{(n-1)w_i}\right) + \frac{c_j}{C},$$

for any user $i \in [n]$ *and link* $j \in [m]$.

Related Links: Fully Mixed Nash Equilibria (2)

Theorem 25. [GLMMS03] *Consider the model of arbitrary users and related links. Then, for any pure Nash equilibrium* $\mathbf{L}$, $\mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{L}) \leq \mathsf{SC}(\mathbf{w}, \mathbf{c}, \mathbf{F})$.

Theorem 26. [LMMRSV03] *Consider the model of two identical users and related links. Then, the Fully Mixed Nash Equilibrium Conjecture is valid.*

Further Variants of the KP-Model

Recently, many variants of the KP-Model were studied. Basically, they alter the definition of latency cost and/or the definition of social cost.

- Restricted Links: In the restricted links model, every user $i \in [n]$ is only allowed to assign his traffic to links in his strategy set $S_i \subseteq [m]$.
  [AART03 , GLMM04a]
- Unrelated Links: In the unrelated links model there exists neither an ordering on the traffics nor on the capacities. We denote by w_{ij} the traffic of user $i \in [n]$ on link $j \in [m]$.
  [AART03]
- Polynomial Social Cost: Social cost is defined as the expectation of the sum of polynomials with non-negative coefficients evaluated at the link latencies.
  [LMMR04 , GLMM04b]
- Convex Latency Functions: Latency functions can be arbitrary convex functions, and social cost is defined as the sum of individual cost.
  [GLMMR04]

Thank you!!!

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