Network Design with Selfish Agents

GI Seminar, Dagstuhl 2004

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Introduction

Literature

- E. Anshelevich, A. Dasgupta, E. Tardos, T. Wexler
  Near-Optimal Network Design with Selfish Agents, STOC 2003
- E. Anshelevich, A. Dasgupta, J. Kleinberg, E. Tardos, T. Wexler, T. Roughgarden
  The Price of Stability for Network Design with Fair Cost Allocation, FOCS 2004
- J. Feigenbaum, C. Papadimitriou, S. Shenker
  Sharing the Cost of Multicast Transmissions, JCSS 2001
- G. Robins, A.Zelikowsky
  Improved Steiner Tree Approximation in Graphs, SODA 2000
- other literature about routing and games

Motivation I

Think of sea transport companies or broadband internet providers

- each company needs to connect a few ports or users
- every connection has a constant cost
- connection is bought if all together pay for it
- no additional utility

Motivation II

Companies (agents) are **selfish**

- we do not consider negotiations, communication
- no external mechanism or regulation
- all desired users must be connected, no tradeoff
- everyone will go for a cheaper price if possible

Formalized Model

Network is modeled as an undirected graph $G = (V, E)$ and there are N agents

- each edge e has a cost $c(e)$
- each agent has a set $V_i \subseteq V$ he needs to connect
- strategy of agent (player) i assigns to edge e payment $p_i(e)$
- play $p = (p_1, \ldots .p_N)$ induces a graph G_p of bought edges

$$e \in G_p \iff \Sigma_i p_i(e) \geq c(e)$$

- play is correct if each set V_i is connected in G_p

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Nash Equilibria

- Total payment of player i is given by

$$P_i = \Sigma_{e \in E}\, p_i(e).$$

- Play p is a Nash equilibrium if p_i minimizes the total payment of player i necessary for a correct play. This must hold for each player i when all p_j remain unchanged for $j \neq i$.
- Play p is an c-approximate Nash equilibrium if each player can decrease the total payment only by a factor of c by choosing a different strategy.

Basic Properties of Nash Equilibria

If p is a Nash equilibrium and T^i the smallest tree in G_p connecting all vertices of player i (i.e. V_i):

- each player i only contributes to costs of edges in T^i
- each edge is either fully paid for or not at all
- G_p is a forest

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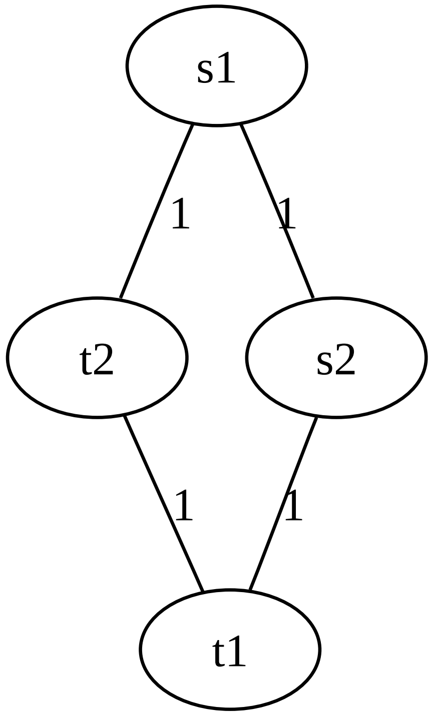
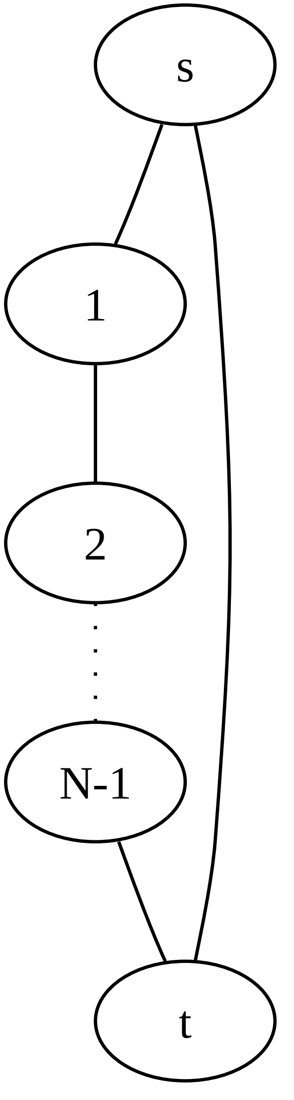
Equilibria Do Not Always Exist

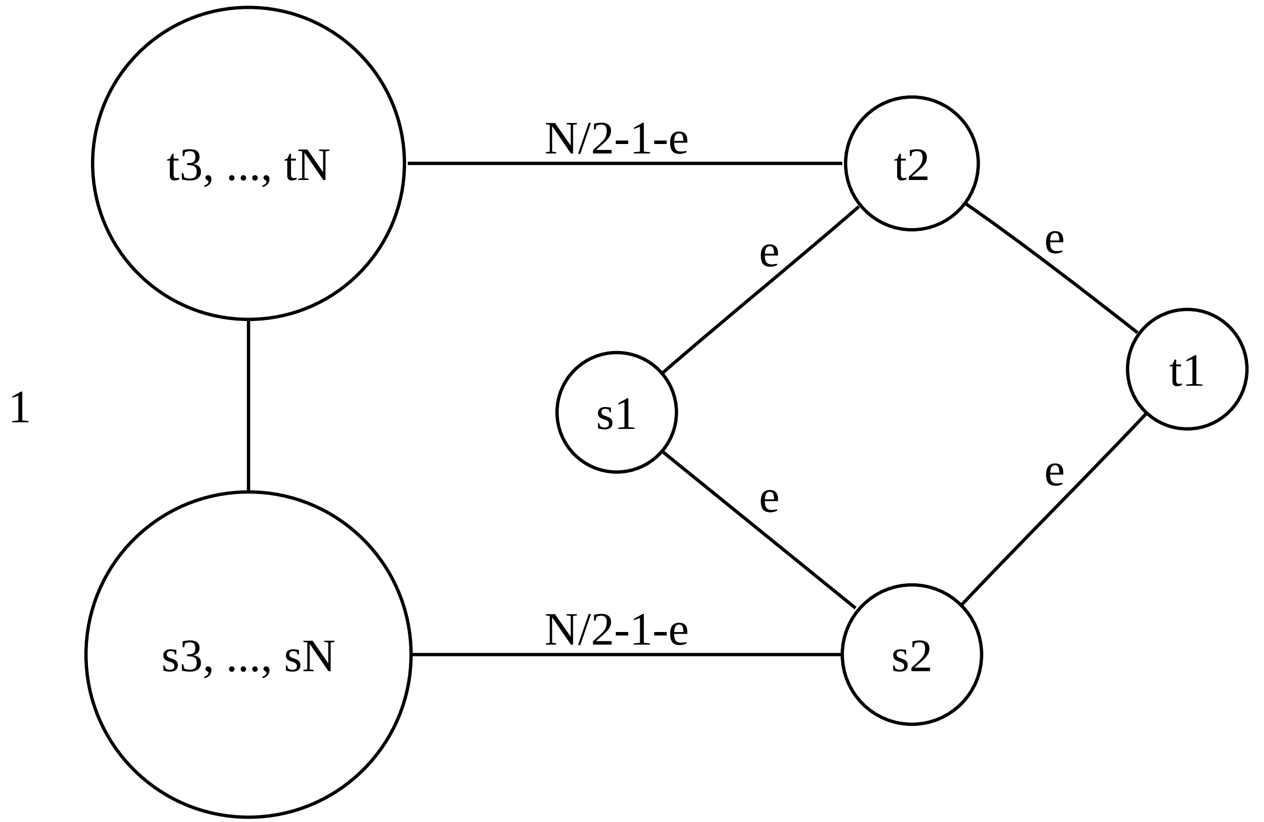
- 

- assume w.l.o.g. that $t_2 - s_1 - s_2 - t_1$ bought
- path $s_2 - t_1$ fully paid for by 1 (not in T^2) and $t_2 - s_1$ by 2 (not in T^1)
- each player can buy $t_1 - t_2$ and spare $s_1 - s_2$

Pessimistic Price of Anarchy N



Optimistic Price of Anarchy



- best cost $1 + 3\epsilon$, best equilibrium cost $N - 2 + \epsilon$
- not possible in single source games

Single Source Connection Games

Single Source Games

- *Definition:* all players share a common vertex s and in addition each player has exactly one other t_i so $V_i = \{s, t_i\}$
- Let T^* be the minimum cost tree rooted at s
- *Theorem:* there is a Nash equilibrium purchasing T^*

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Notation

We will be going in reverse BFS order through T^* and set payments.

- T_e is a subtree of T^* disconnected from s when e is removed
- $p_i(T^*) = \Sigma_{e \in T^*} p_i(e)$
- $p(e) = \Sigma_i p_i(e)$
- **modified costs** for player i
  - $c'(e) = p_i(e)$ for all $e \in T^*$
  - $c'(e) = c(e)$ for all $e \notin T^*$

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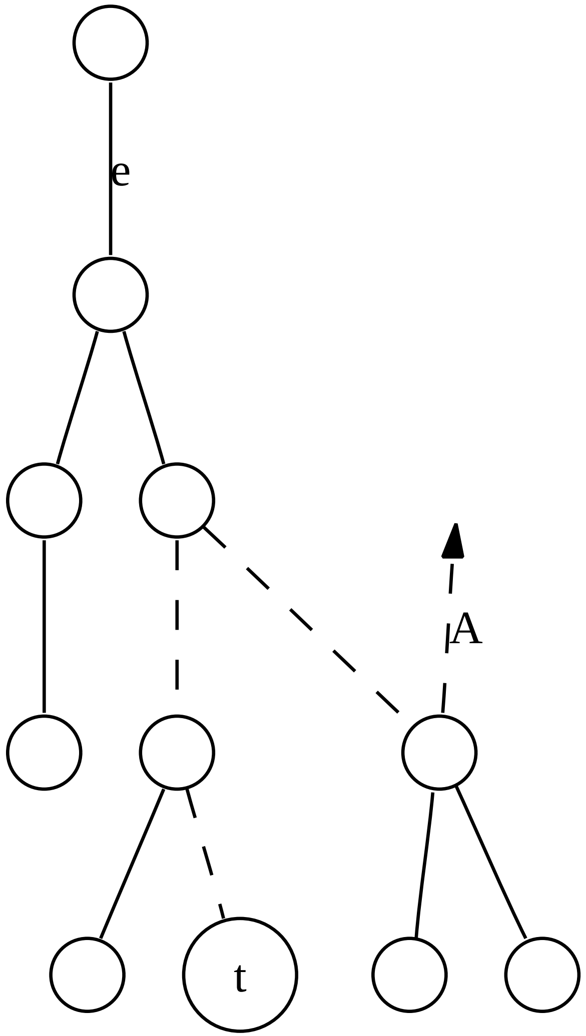
Algorithm

- Initialize $p_i(e) = 0$ for all players and edges
- Loop through all edges e in T^* in reverse BFS order
  - Loop through all players i with $t_i \in T_e$ until e paid for
    - If e is a cut in G set $p_i(e) = c(e)$
    - Else let χ_i be the cost of the cheapest path A from s to t_i in $G \setminus \{e\}$ under the modified costs,
    - Set $p_i(e) = \min\{\chi_i - p_i(T^*), c(e) - p(e)\}$

Overview



Correctness I

The above algorithm returns a Nash equilibrium if it terminates. Consider at some stage determining i's payment to e

- updated costs reflect the costs i faces if he deviates
- updated costs never exceed total payment
- at no stage no player has incentive to deviate

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Correctness II

The above algorithm succeeds to pay for each edge. Assume at some point players with vertices in T_e are unwilling to pay for e. For each player i with $V_i \cap T_e \neq \emptyset$ we have an alternate path A_i explaining his payment

- in the algorithm $\chi_i = c'(A_i)$
- if more such paths exist, choose one including as many ancestors of t_i as possible
- we will build a cheaper tree than T^* from these paths

Path Lemma I

Lemma: When A_i is an alternate path then it has precisely three segments, one in T_e, one in $E \setminus T^*$ and the rest in $T^* \setminus T_e$.

- once A_i reaches $T^* \setminus T_e$ it remains there as edges there have cost $c'(f) = p_i(f) = 0$ (reverse BFS)
- suppose that it leaves T_e and returns to T_e
- let P_1 be the starting segment in T_e and P_2 continue outside T_e until a vertex x back in T_e
- consider y - the lowest common ancestor of x and t_i

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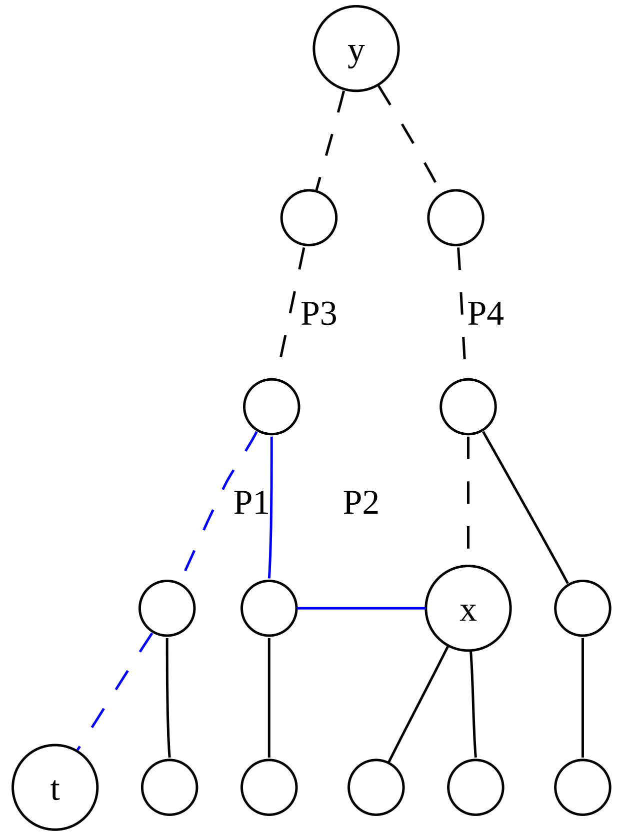
Path Lemma II



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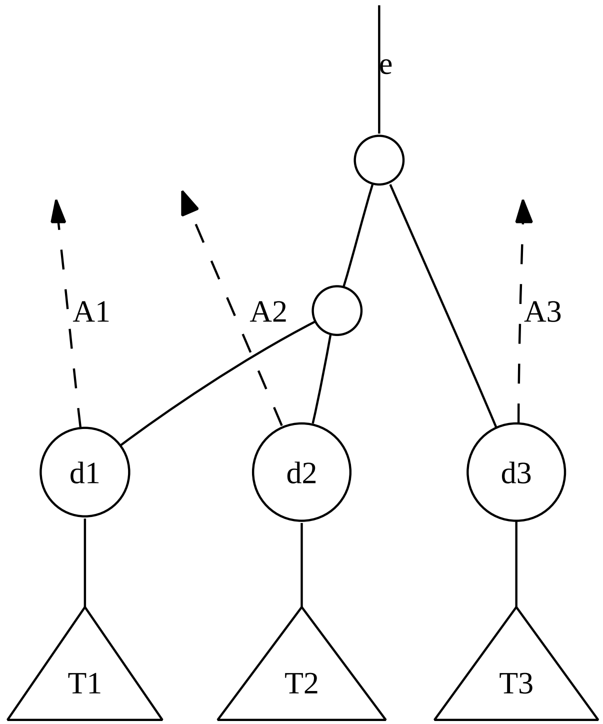
- e is over y and the algorithm did not fail before
- $P_3 \cup P_4$ is at least as cheap as $P_1 \cup P_2$ and induces a higher ancestor

Correctness Proof



- let each player with vertices in T_e deviate and buy the alternate path instead of e
- $T^* \setminus e$ but with these paths is cheaper than T^*

Approximate Algorithm

- Optimal cost tree is NP-hard to compute, $1,55$-approximation exists.
- Given a α-approximate tree T_α we have a polynomial algorithm in $\frac{1}{\epsilon}$ for $(1+\epsilon)$-approximate equilibrium with cost at most $c(T_\alpha)$.
- The idea is to use the alternate paths to build a better tree if some edge is not paid for.
- To make only substantial improvement we use the possibility to deviate by a factor $(1+\epsilon)$.

Results

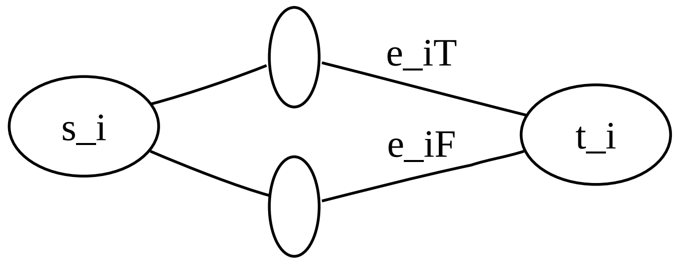
- Optimistic price of anarchy in single source connection games is 1.
- The same holds for directed graphs with extended arguments (path lemma).
- Extension when each player has a maximal cost that can not be exceeded reduces to the directed case.
- Allowing (1+ϵ)-approximate equilibria makes it possible to use approximate cost tree.

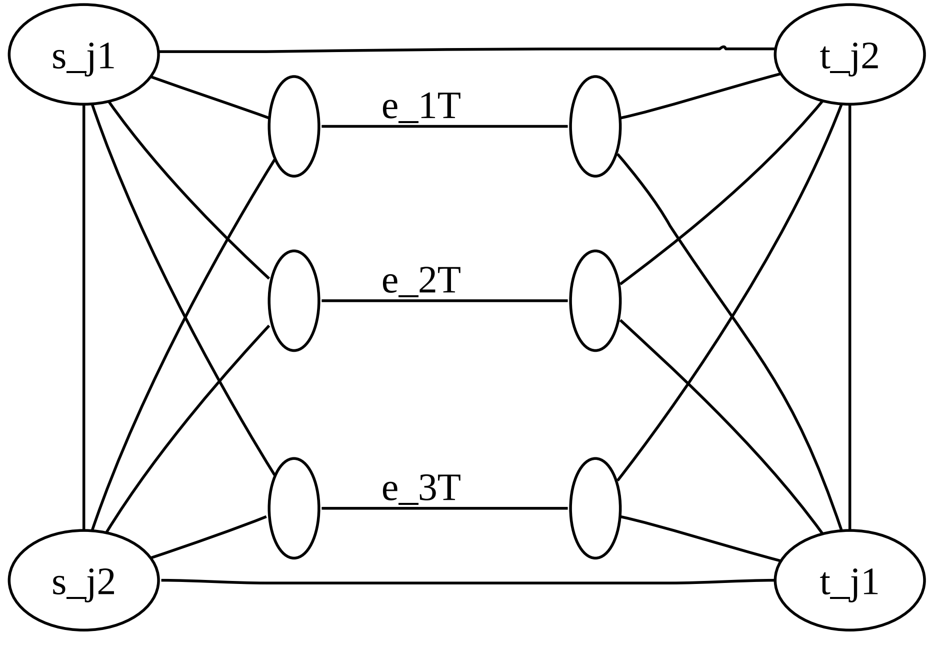
Multiple Source Connection Games

Situation in General Games

- Optimistic price of anarchy near the upper bound where the player can buy the whole optimal tree himself.
- Existence of Nash equilibrium is itself NP-hard (3-SAT reduction).
- But a 3-approximate Nash equilibrium always exists.

Equilibrium Existence NP-hard





3-approximate Nash Equilibrium

- We will show a payment scheme for T^*.
- Lower bound used for approximation: cost of connection set.
- Connection set S for player i is a subset of edges of T^i such that for each connected component of $T^* \setminus S$:
  - player i has a vertex in C, or
  - if j has a vertex in C then he has all vertices in C.
- Intuition: after removing S if i has connected his vertices somehow, then all other players have their still connected.

Lower Bound Lemma

Let p be a play with players purchasing T^* which obeys the following.

- If $p(e) > 0$ then e is bought fully by a single player.
- Each player i buys only edges in his tree T^i.
- The set of edges each player buys is a union of at most α connection sets.

Then p is an α-approximate Nash equilibrium.

Sketch of Payment Scheme

- First purchase all edges in T_i that are not in any other T_j (clearly at most one connection set).
- Contract the purchased edges. Vertices will pay instead of players for simplicity.
- Select an arbitrary path between vertices of the same player, direct it and start with it as an element of path set.
- For each path in the set determine which vertices pay for which edges and extend the path set.

Sketch of Path Payment I

- For each vertex v_k on the path consider the subtree of T^* rooted at v_k obtained by cutting the edges that belong to the path.
- Let S_k be the set of vertices in this subtree that belong to some player i who also has some vertices outside the subtree.
- Assign the part of the path to the next point where player i can reach a subtree with his vertex to the vertex from S_k.
- Consider the start and end of the path as special cases.

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Sketch of Path Payment II

- *Define* a link to be a maximal set of edges on the path such that if an edge in the link was assigned to some vertex t then the other edges have also been assigned to t.
- Link is really a connection set of any vertex to which the edges of the link can be assigned.
- We need to choose exactly one vertex from each set S_k and have these pay for the whole path with each one paying for at most one link.
- Create an abstract bipartite graph with links as nodes on one side and vertices as nodes on the other side and use Hall's theorem to choose a vertex from each S_k to pay for one link.

Correctness of Algorithm

- Sum the links chosen for a vertex.
- Sum over vertices of a player.
- This will not exceed two full connection sets.
- Adding the connection set before contraction and using the lower bound lemma we get a 3-approximate equilibrium.

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Approximate Algorithm

- Try to use the ideas shown before to obtain an $(3+\epsilon)$-approximate equilibrium from an α-approximate tree.
- We end with at most 3 connection sets for each player.
- Such algorithm requires an optimal cost tree finder as a subroutine to check if each connection set is cheaper than the cheapest deviation.
- Then we get an $(3+\epsilon)$-approximate equilibrium or iterate correcting α.
- In polynomial time we can only guarantee $(4,55+\epsilon)$-approximate equilibrium.

Results

- Establishing if there exists an equilibrium is NP-hard, even optimistic price of anarchy is high.
- In the special case of game with two players each with two vertices the equilibrium can be found in polynomial time.
- 3-approximate equilibria exist, $(4,55+\epsilon)$-approximate equilibrium for $1,55$-approximate cost tree can be found in polynomial time.
- Lower bound for approximate Nash equilibria is $\frac{3}{2}$, there exist a sequence of games that any equilibrium purchasing the optimal solution must be (in the limit) $(\frac{3}{2})$-approximate.

Fair Connection Games

- Each player only chooses a path to connect each pair of nodes
- Price determined using Shapley cost-sharing mechanism
  - cost of each edge shared equally by all players using it
- Pure Nash equilibrium always exists
- Total cost paid in this way by the best Nash equilibrium is at most factor $\log(N)$ worse than optimal

Conclusions

- The price of anarchy in connection games is high.
- Deciding if an equilibrium exists is in general NP-hard.
- In single source games it is much better, equilibrium always exists and optimistic price of anarchy is 1.
- With approximate equilibria it is much better.
  - In single source games we can find the equilibria on approximate trees in polynomial time.
  - In general games 3-approximate equilibria exist and $(4, 55 + \epsilon)$-approximate can be found on approximate cost trees in polynomial time.

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Thank you!