Network Design with Selfish Agents

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Introduction
Literature

- E. Anshelevich, A. Dasgupta, E. Tardos, T. Wexler
  *Near-Optimal Network Design with Selfish Agents*, STOC 2003

  *The Price of Stability for Network Design with Fair Cost Allocation*, FOCS 2004

- J. Feigenbaum, C. Papadimitriou, S. Shenker
  *Sharing the Cost of Multicast Transmissions*, JCSS 2001

- G. Robins, A. Zelikowsky
  *Improved Steiner Tree Approximation in Graphs*, SODA 2000

- other literature about routing and games
Motivation I

Think of sea transport companies or broadband internet providers

- each company needs to connect a few ports or users
- every connection has a constant cost
- connection is bought if all together pay for it
- no additional utility
Motivation II

Companies (agents) are selfish

- we do not consider negotiations, communication
- no external mechanism or regulation
- all desired users must be connected, no tradeoff
- everyone will go for a cheaper price if possible
Formalized Model

Network is modeled as an undirected graph $G = (V, E)$ and there are $N$ agents

- each edge $e$ has a cost $c(e)$
- each agent has a set $V_i \subseteq V$ he needs to connect
- strategy of agent (player) $i$ assigns to edge $e$ payment $p_i(e)$
- play $p = (p_1, \ldots, p_N)$ induces a graph $G_p$ of bought edges

$$e \in G_p \iff \sum_i p_i(e) \geq c(e)$$

- play is correct if each set $V_i$ is connected in $G_p$
Nash Equilibria

- Total payment of player $i$ is given by

$$P_i = \sum_{e \in E} p_i(e).$$

- Play $p$ is a Nash equilibrium if $p_i$ minimizes the total payment of player $i$ necessary for a correct play. This must hold for each player $i$ when all $p_j$ remain unchanged for $j \neq i$.

- Play $p$ is an $c$-approximate Nash equilibrium if each player can decrease the total payment only by a factor of $c$ by choosing a different strategy.
Basic Properties of Nash Equilibria

If $p$ is a Nash equilibrium and $T^i$ the smallest tree in $G_p$ connecting all vertices of player $i$ (i.e. $V_i$):

- each player $i$ only contributes to costs of edges in $T^i$
- each edge is either fully paid for or not at all
- $G_p$ is a forest
assume w.l.o.g. that $t_2 - s_1 - s_2 - t_1$ bought

path $s_2 - t_1$ fully paid for by 1 (not in $T^2$) and $t_2 - s_1$ by 2 (not in $T^1$)

each player can buy $t_1 - t_2$ and spare $s_1 - s_2$
Pessimistic Price of Anarchy $N$
Optimistic Price of Anarchy

- best cost $1 + 3\varepsilon$, best equilibrium cost $N - 2 + \varepsilon$
- not possible in single source games
Single Source Connection
Games
Definition: all players share a common vertex $s$ and in addition each player has exactly one other $t_i$ so $V_i = \{s, t_i\}$

Let $T^*$ be the minimum cost tree rooted at $s$

Theorem: there is a Nash equilibrium purchasing $T^*$
Notation

We will be going in reverse BFS order through $T^*$ and set payments.

- $T_e$ is a subtree of $T^*$ disconnected from $s$ when $e$ is removed
- $p_i(T^*) = \sum_{e \in T^*} p_i(e)$
- $p(e) = \sum_i p_i(e)$
- **modified costs** for player $i$
  - $c_i'(e) = p_i(e)$ for all $e \in T^*$
  - $c_i'(e) = c(e)$ for all $e \notin T^*$
Algorithm

1. Initialize $p_i(e) = 0$ for all players and edges
2. Loop through all edges $e$ in $T^*$ in reverse BFS order
   1. Loop through all players $i$ with $t_i \in T_e$ until $e$ paid for
      1. If $e$ is a cut in $G$ set $p_i(e) = c(e)$
      2. Else let $\chi_i$ be the cost of the cheapest path $A$
         from $s$ to $t_i$ in $G \setminus \{e\}$ under the modified costs,
      3. Set $p_i(e) = \min\{\chi_i - p_i(T^*), c(e) - p(e)\}$
Overview
Correctness I

The above algorithm returns a Nash equilibrium if it terminates. Consider at some stage determining $i$’s payment to $e$

- updated costs reflect the costs $i$ faces if he deviates
- updated costs never exceed total payment
- at no stage no player has incentive to deviate
Correctness II

The above algorithm succeeds to pay for each edge. Assume at some point players with vertices in $T_e$ are unwilling to pay for $e$. For each player $i$ with $V_i \cap T_e \neq \emptyset$ we have an alternate path $A_i$ explaining his payment

- in the algorithm $\chi_i = c'(A_i)$
- if more such paths exist, choose one including as many ancestors of $t_i$ as possible
- we will build a cheaper tree than $T^*$ from these paths
Path Lemma I

Lemma: When $A_i$ is an alternate path then it has precisely three segments, one in $T_e$, one in $E \setminus T^*$ and the rest in $T^* \setminus T_e$.

- once $A_i$ reaches $T^* \setminus T_e$ it remains there as edges there have cost $c'(f) = p_i(f) = 0$ (reverse BFS)
- suppose that it leaves $T_e$ and returns to $T_e$
- let $P_1$ be the starting segment in $T_e$ and $P_2$ continue outside $T_e$ until a vertex $x$ back in $T_e$
- consider $y$ - the lowest common ancestor of $x$ and $t_i$
Path Lemma II

\[ e \] is over \( y \) and the algorithm did not fail before

\[ P_3 \cup P_4 \] is at least as cheap as \( P_1 \cup P_2 \) and induces a higher ancestor
Correctness Proof

Let each player with vertices in $T_e$ deviate and buy the alternate path instead of $e$

$T^* \setminus e$ but with these paths is cheaper than $T^*$
Approximate Algorithm

- Optimal cost tree is NP-hard to compute, 1, 55-approximation exists.

- Given a $\alpha$-approximate tree $T_\alpha$ we have a polynomial algorithm in $\frac{1}{\epsilon}$ for $(1 + \epsilon)$-approximate equilibrium with cost at most $c(T_\alpha)$.

- The idea is to use the alternate paths to build a better tree if some edge is not paid for.

- To make only substantial improvement we use the possibility to deviate by a factor $(1 + \epsilon)$. 
Results

- Optimistic price of anarchy in single source connection games is $1$.
- The same holds for directed graphs with extended arguments (path lemma).
- Extension when each player has a maximal cost that can not be exceeded reduces to the directed case.
- Allowing $(1+\epsilon)$-approximate equilibria makes it possible to use approximate cost tree.
Multiple Source Connection Games
Situation in General Games

- Optimistic price of anarchy near the upper bound where the player can buy the whole optimal tree himself.
- Existence of Nash equilibrium is itself NP-hard (3-SAT reduction).
- But a 3-approximate Nash equilibrium always exists.
Equilibrium Existence NP-hard
3-approximate Nash Equilibrium

We will show a payment scheme for $T^*$. Lower bound used for approximation: cost of connection set.

Connection set $S$ for player $i$ is a subset of edges of $T^i$ such that for each connected component of $T^* \setminus S$:

- player $i$ has a vertex in $C$, or
- if $j$ has a vertex in $C$ then he has all vertices in $C$.

Intuition: after removing $S$ if $i$ has connected his vertices somehow, then all other players have their still connected.
Lower Bound Lemma

Let \( p \) be a play with players purchasing \( T^* \) which obeys the following.

- If \( p(e) > 0 \) then \( e \) is bought fully by a single player.
- Each player \( i \) buys only edges in his tree \( T^i \).
- The set of edges each player buys is a union of at most \( \alpha \) connection sets.

Then \( p \) is an \( \alpha \)-approximate Nash equilibrium.
Sketch of Payment Scheme

- First purchase all edges in $T_i$ that are not in any other $T_j$ (clearly at most one connection set).
- Contract the purchased edges. Vertices will pay instead of players for simplicity.
- Select an arbitrary path between vertices of the same player, direct it and start with it as an element of path set.
- For each path in the set determine which vertices pay for which edges and extend the path set.
Sketch of Path Payment I

- For each vertex $v_k$ on the path consider the subtree of $T^*$ rooted at $v_k$ obtained by cutting the edges that belong to the path.
- Let $S_k$ be the set of vertices in this subtree that belong to some player $i$ who also has some vertices outside the subtree.
- Assign the part of the path to the next point where player $i$ can reach a subtree with his vertex to the vertex from $S_k$.
- Consider the start and end of the path as special cases.
Sketch of Path Payment II

- Define a link to be a maximal set of edges on the path such that if an edge in the link was assigned to some vertex $t$ then the other edges have also been assigned to $t$.

- Link is really a connection set of any vertex to which the edges of the link can be assigned.

- We need to choose exactly one vertex from each set $S_k$ and have these pay for the whole path with each one paying for at most one link.

- Create an abstract bipartite graph with links as nodes on one side and vertices as nodes on the other side and use Hall’s theorem to choose a vertex from each $S_k$ to pay for one link.
Correctness of Algorithm

- Sum the links chosen for a vertex.
- Sum over vertices of a player.
- This will not exceed two full connection sets.
- Adding the connection set before contraction and using the lower bound lemma we get a $3$-approximate equilibrium.
Approximate Algorithm

- Try to use the ideas shown before to obtain an $(3 + \epsilon)$-approximate equilibrium from an $\alpha$-approximate tree.
- We end with at most 3 connection sets for each player.
- Such algorithm requires an optimal cost tree finder as a subroutine to check if each connection set is cheaper than the cheapest deviation.
- Then we get an $(3 + \epsilon)$-approximate equilibrium or iterate correcting $\alpha$.
- In polynomial time we can only guarantee $(4, 55 + \epsilon)$-approximate equilibrium.
Results

- Establishing if there exists an equilibrium is NP-hard, even optimistic price of anarchy is high.

- In the special case of game with two players each with two vertices the equilibrium can be found in polynomial time.

- 3-approximate equilibria exist, $(4, 55 + \epsilon)$-approximate equilibrium for $1, 55$-approximate cost tree can be found in polynomial time.

- Lower bound for approximate Nash equilibria is $\frac{3}{2}$, there exist a sequence of games that any equilibrium purchasing the optimal solution must be (in the limit) $(\frac{3}{2})$-approximate.
Fair Connection Games

- Each player only chooses a path to connect each pair of nodes
- Price determined using Shapley cost-sharing mechanism
  - cost of each edge shared equally by all players using it
- Pure Nash equilibrium always exists
- Total cost paid in this way by the best Nash equilibrium is at most factor $\log(N)$ worse than optimal
Conclusions

- The price of anarchy in connection games is high.
- Deciding if an equilibrium exists is in general NP-hard.
- In single source games it is much better, equilibrium always exists and optimistic price of anarchy is 1.
- With approximate equilibria it is much better.
  - In single source games we can find the equilibria on approximate trees in polynomial time.
  - In general games 3-approximate equilibria exist and \((4, 55 + \epsilon)\)-approximate can be found on approximate cost trees in polynomial time.
Thank you!