Network Bandwidth Allocation

GI-Dagstuhl-Seminar:

Game-Theoretic Analyses of the Internet

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Motivation

Task:

Charging and rate control for network resources (e.g. ATM networks offering available bit rate service).

Allocate capacities / bandwidths efficiently among users.

Model:

- Users submit bids for each link (willingness-to-pay per unit).
- Then a network manager determines the price of each link and responds the allocated rates (for each link and user).
  We assume *proportional allocation* (rate in proportion to bid; corresponds to a "raffle" in economics) and no *price discrimination* (manager treats all users alike).
- Utility for user: monetary function depending on the allocated aggregate rate.

References

Talk is based on:

- Frank P. Kelly, *Charging and rate control for elastic traffic*, European Transactions on Telecommunications, vol. 8 (1997).
- Ramesh Johari, John N. Tsitsiklis, *Efficiency loss in a network resource allocation game*, to appear in Mathematics of Operations Research (2004).

The Goal

Two assumptions about user's behavior:

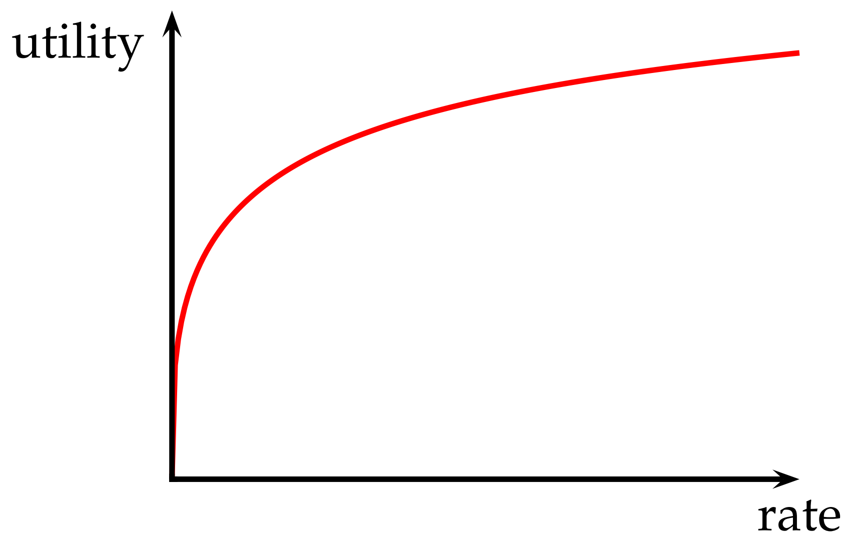
- *Price taking:* Users do not anticipate effect of their bidding on the price.
- *Price anticipating:* They do so. Due to the selfish behavior the model becomes a game.

Questions:

- How can we maximize the aggregate utility / social optimum?
- What appropriate notion of equilibria do we have?
- When do equilibria exist?
- Does anticipating behavior significantly worsen performance? Can we quantify the loss of efficiency ("price of anarchy")?

Utility Functions

Restriction: *strictly increasing*, *continuously differentiable* and *concave* utility functions $U_r : \mathbb{R}_{\geq 0} \to \mathbb{R}$ for each user r.



Shenker: Concavity corresponds to *elastic traffic* (tolerance of delays, decreasing marginal degradation due to incremental increases in delays, e.g. file transfer, email etc.)

Allows the use of techniques of non-linear programming / convex optimization.

Overview

First, we consider a *single link* with capacity $C > 0$.

Users $1, \ldots, R$. User r submits bid $w_r \geq 0$ (charge per unit time).

Manager chooses rate allocation $d_r \geq 0$ for user r *in proportion* to w_r (rate d_r is a fraction of C, flow per unit time).

Price taking users (Kelly, 1997):

- There exists a *competitive equilibrium*. It is unique on condition that each U_r is *strictly concave*.
- It corresponds to a maximum possible aggregate utility, i.e. to a solution of the optimization problem

  (System) Maximize $\sum_r U_r(d_r)$ subject to $\sum_r d_r \leq C$ and $d_r \geq 0$.
- In particular, the problem can be decomposed into $R + 1$ subsidiary problems: one for each user and one for the network.

Overview II

Price anticipating users (Hajek, Gopalakrishnan, 2002):

- At least *two* users: There exists a unique (pure) *Nash equilibrium*.
- It is characterized by an optimization problem in the style of (System) with modified utility functions.
- An equilibrium may not exist for a *single* user (due to a discontinuity at a zero bid).

Price of anarchy (Johari, Tsitsiklis, 2004):

- Aggregate utility corresponding to NE no worse than $\frac{3}{4}$ the optimal aggregate utility corresponding to (System).

  Selfish behavior of users versus social optimum: Efficiency loss of no more than 25%.

Overview III: Extension to General Networks

- A model with alternative routing.
- Users submit individual bids for each link.
- Users send maximum flow possible (after receiving the allocated rates from the manager).

Bottleneck problem for natural extension of single link model:
NE may not exist, due to a discontinuity in the payoff.

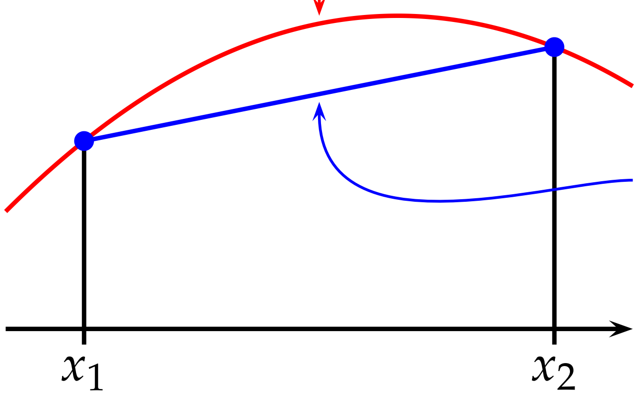
Extended game to fix this problem: Users not only submit bids, but also *rate requests*. Requests are only taken into account if total bidding to a link is zero.

Now all results extend to general networks:

- Existence of NE (even if $R = 1$).
- Equilibria of original game correspond to equil. of extended game.
- Price of anarchy: Efficiency loss of no more than 25%.
- In fact: restrictions to utility functions can be relaxed.

Repetition: Concave Functions

- Fix $S \subseteq \mathbb{R}^n$ non-empty & convex. $f : S \to \mathbb{R}$ *concave* if
$$f(\lambda \cdot x_1 + (1-\lambda) \cdot x_2) \geq \lambda \cdot f(x_1) + (1-\lambda) \cdot f(x_2)$$
for all $x_1, x_2 \in S$ and $\lambda \in (0,1)$; *strictly* if $>$ for $x_1 \neq x_2$.



- f concave: any local maximum is already global (in S);
  f strictly concave: maximum is unique.
- f concave & differentiable: x_0 maximum iff $\nabla f(x_0)^t \cdot (x - x_0) \leq 0$ for all $x \in S$.
- If $S = \mathbb{R}^n_{\geq 0}$: x_0 maximum iff $\frac{\partial f}{\partial x_i}(x_0) \begin{cases} = 0 & \text{, if } x_0 > 0 \\ \leq 0 & \text{, if } x_0 = 0 \end{cases}$ for all i.

Part 1

Price Taking Users

Notation

- Single link with capacity $C > 0$; users $1, \ldots, R$.
- User r submit bid $w_r \geq 0$; $\mathbf{w} = (w_1, \ldots, w_R)$.
- User r receives rate allocation $d_r \geq 0$ from the manager; $\mathbf{d} = (d_1, \ldots, d_R)$.
- Utility functions U_r with domain $d_r \geq 0$; strictly increasing, continuously differentiable, and concave.

Aggregate utility: $U(\mathbf{d}) := \sum_r U_r(d_r)$.

Social optimum:

(System) Maximize $U(\mathbf{d})$ subject to $\sum_r d_r \leq C$ and $d_r \geq 0$.

Pricing Scheme, Payoffs, and Competitive Equilibrium

In general, U not available for manager $\rightsquigarrow$ need for pricing scheme.

Assumption: Proportional allocation, no price discrimination, entire capacity is allocated.

Price: Manager determines price (per unit flow) μ and rate allocation

$$\mu = \frac{\sum_r w_r}{C} \quad \text{and} \quad \mathbf{d} = \begin{cases} \frac{1}{\mu}\mathbf{w} & \text{, if } \mathbf{w} \neq 0 \\ 0 & \text{, otherwise.} \end{cases}$$

User's payoff: Given price $\mu > 0$ as parameter:

$$P_r(w_r; \mu) = U_r\left(\frac{w_r}{\mu}\right) - w_r.$$

Typical: Payoff quasi-linear in money.

Definition: A pair $(\mathbf{w}, \mu)$ with $\mu > 0$ is a *competitive equilibrium* if

- Users maximize their payoff, i.e. for each $r = 1, \ldots, R$:

$$\forall\, \overline{w}_r \geq 0 : P_r(w_r; \mu) \geq P_r(\overline{w}_r; \mu).$$

- Manager "clears the market": $\mu = \frac{\sum_r w_r}{C}$.

Theorem (Kelly):

- There exists a competitive equilibrium $(\mathbf{w}, \mu)$.
- Scalar μ (price) is unique.
- Vector $\mathbf{d} = \frac{1}{\mu}\mathbf{w}$ is a solution to (System).
- If each U_r is strictly concave then $\mathbf{w}$ is unique as well.

Recall: This is *not* a game.

Interpretation: (System) can be decomposed into $R + 1$ subsidiary problems, one for each user and one for the network, where price μ mediates between the problems.

Proof sketch:

- U_r concave $\rightsquigarrow$ P_r concave. Given $\mu > 0$, w_r maximizes P_r iff

$$P'_r(w_r;\mu) \begin{cases} = 0 & \text{, if } w_r > 0 \\ \leq 0 & \text{, if } w_r = 0 \end{cases} \quad \text{iff} \quad U'_r\left(\frac{w_r}{\mu}\right) \begin{cases} = \mu & \text{, if } w_r > 0 \\ \leq \mu & \text{, if } w_r = 0. \end{cases} \tag{1}$$

- Lagrangian for (System): $\mathcal{L}(\mathbf{d},\lambda) = U(\mathbf{d}) - \lambda \cdot (\sum_r d_r - C)$ with Lagrange multiplier $\lambda \geq 0$ (penalty for capacity constraint; economic interpretation: Shadow price of additional capacity).

Slater constraint qualification at $\mathbf{d} = 0$: $0 = \sum_r d_r < C$.

Stationary conditions (concave objective fct. & convex feasible region): Feasible $\mathbf{d}$ is optimal iff there is $\lambda \geq 0$ such that for all r

$$\frac{\partial \mathcal{L}}{\partial d_r}(\mathbf{d},\lambda) \begin{cases} = 0 & \text{, if } d_r > 0 \\ \leq 0 & \text{, if } d_r = 0 \end{cases} \quad \text{iff} \quad U'_r(d_r) \begin{cases} = \lambda & \text{, if } d_r > 0 \\ \leq \lambda & \text{, if } d_r = 0. \end{cases} \tag{2}$$

- Each U_r continuous, hence U continuous; feasible region compact $\rightsquigarrow$ there exists at least one optimal solution $\mathbf{d}$ to (System).

Thus: there is $\lambda \geq 0$ satisfying (2).

- U strictly increasing $\rightsquigarrow \sum_r d_r = C > 0 \rightsquigarrow$ at least one $0 < d_r \rightsquigarrow$ $0 < U_r'(d_r) = \lambda$.

By comparing (1) and (2): $\lambda = \mu$ and $\mathbf{d} = \frac{1}{\mu}\mathbf{w}$.

- μ unique:

Otherwise $(\mathbf{d}, \mu)$ and $(\overline{\mathbf{d}}, \overline{\mu})$ with $\mu < \overline{\mu}$, both satisfying (2).

U_r concave $\rightsquigarrow U_r'$ non-increasing. Hence, for $\overline{d}_r > 0$:
$U_r'(d_r) \leq \mu < \overline{\mu} = U_r'(\overline{d}_r)$ implies $\overline{d}_r < d_r$.

At least one $\overline{d}_r > 0 \rightsquigarrow C = \sum_r \overline{d}_r < \sum_r d_r = C$. Contradiction.

- Since feasible region is convex: If each U_r is strictly concave, then U is strictly concave and solution $\mathbf{d}$ to (System) is unique
$\rightsquigarrow$ $\mathbf{w}$ is unique. □

Part 2

Price Anticipating Users

Payoffs and Nash Equilibrium

Price anticipating users realize how μ is determined by the manager and adjust their payoff accordingly $\rightsquigarrow$ model becomes a *game*.

User's payoff: Let vector $\mathbf{w}_{-r}$ be vector $\mathbf{w}$ without component w_r. Given $\mathbf{w}_{-r}$ as parameter:

$$Q_r(w_r; \mathbf{w}_{-r}) = \begin{cases} U_r\left(\frac{w_r}{\sum_s w_s} \cdot C\right) - w_r & \text{, if } w_r > 0 \\ U_r(0) & \text{, if } w_r = 0. \end{cases}$$

- May be discontinuous at $w_r = 0$, if $\mathbf{w}_{-r} = 0$.

Definition: Vector $\mathbf{w} \geq 0$ is a (pure) *Nash equilibrium*, NE, if users maximize their payoff *given* the other users' bids, i.e. for all r:

$$\forall\, \overline{w}_r \geq 0 : Q_r(w_r; \mathbf{w}_{-r}) \geq Q_r(\overline{w}_r; \mathbf{w}_{-r}).$$

Example 1: Single User Game

Discontinuity at 0 may preclude NE.

Single user $\rightsquigarrow$ any positive bid results in entire capacity:

$$Q_1(w) = \begin{cases} U_1(C) - w & \text{, if } w > 0 \\ U_1(0) & \text{, if } w = 0. \end{cases}$$

U_1 strictly increasing $\rightsquigarrow$ $U_1(0) < U_1(C)$.

Assume there is NE w.

- $w = 0$: any $0 < \overline{w} < U_1(C) - U_1(0)$ is profitable deviation: $Q_1(\overline{w}) > Q_1(w)$.
- $w > 0$: any $0 < \overline{w} < w$ leaves rate allocation unchanged, while reducing the payment.

Contradiction.

Modified Utility Functions and Main Result

For given utility functions U_r define

$$U_r^*(d_r) = \left(1 - \frac{d_r}{C}\right) \cdot U_r(d_r) + \left(\frac{d_r}{C}\right) \cdot \left(\frac{1}{d_r} \int_0^{d_r} U_r(z)\, \mathrm{d}z\right).$$

Intuition: $U_r^*(d_r)$ = expectation of U_r w.r.t. probability distribution: place mass of $1 - \frac{d_r}{C}$ on event d_r and uniformly distribute remaining mass on interval $[0, d_r]$.

Aggregate utility $U^*(\mathbf{d}) := \sum_r U_r^*(d_r)$. (System)-style problem:

(Game) Maximize $U^*(\mathbf{d})$ subject to $\sum_r d_r \leq C$ and $d_r \geq 0$.

Theorem (Hajek-Gopalakrishnan): At least two users.

- There exists a *unique* NE $\mathbf{w}$ and it holds $\sum_r w_r > 0$.
- Vector $\mathbf{d}$ with $d_r = \frac{w_r}{\sum_s w_s} \cdot C$ is *unique* solution to (Game).

Proof sketch:

$\mathbf{w}$ NE $\rightsquigarrow$ two components positive (same argument as for single user).

Then: Each $w_r \mapsto \frac{w_r}{w_r+z}$ with $z > 0$: s.c./s.i./c.d. $\rightsquigarrow$ Q_r s.c./c.d. $\rightsquigarrow$ w_r unique maximum of Q_r & satisfies first order optimality condition.

Thus:

- $\mathbf{w}$ NE iff at least two components of $\mathbf{w}$ positive and

$$\frac{\partial Q}{\partial w_r}(w_r, ; \mathbf{w}_{-r}) \begin{cases} = 0 & \text{, if } w_r > 0 \\ \leq 0 & \text{, if } w_r = 0 \end{cases} \quad \text{for each } r. \tag{3}$$

- (3) is equivalent to

$$\begin{aligned} U_r'\left(\frac{w_r}{\sum_s w_s} \cdot C\right)\left(1 - \frac{w_r}{\sum_s w_s}\right) &= \frac{\sum_s w_s}{C}, \quad \text{if } w_r > 0 \\ U_r'(0) &\leq \frac{\sum_s w_s}{C}, \quad \text{if } w_r = 0. \end{aligned} \tag{4}$$

- U_r^* s.c./s.i./c.d. over $0 \le d_r \le C$ with

$$(U_r^*)'(d_r) = U_r'(d_r) \cdot \left(1 - \frac{d_r}{C}\right).$$

As in previous theorem: Consider Lagrangian $\mathcal{L}$ for (Game) with multiplier $\lambda \ge 0$.

- (Game) has unique solution $\mathbf{d}$ (objective function cont. & s.c.; feasible region compact & convex). Thus λ exists.

- At least two components of $\mathbf{d}$ are positive; $\lambda > 0$ and unique.

- By comparing stationary conditions for $\mathcal{L}$ with condition (4):

$$\lambda = \frac{\sum_s w_s}{C} \quad \text{and} \quad d_r = \frac{w_r}{\sum_s w_s} \cdot C$$

and $\mathbf{w} = \lambda \cdot \mathbf{d}$ is NE.

- Reverse argument: Different NE lead to different solutions to (Game). Hence NE unique. □

Part 3

Price of Anarchy

Preliminaries

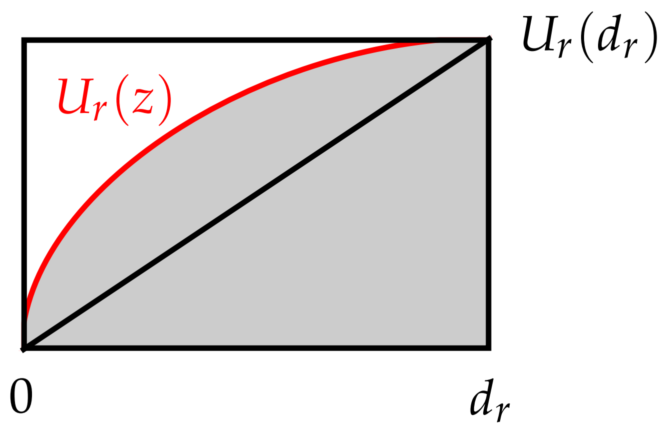
Assume at least two users ($R > 1$) and $U_r(0) \geq 0$ for all r. Let

- $\mathbf{d}^S$ an optimal solution to (System) [a social optimum],
- $\mathbf{d}^G$ the unique optimal solution to (Game) [$\sim$ Nash equilibrium].

By definition of (System): $0 \leq U(d_r^G) \leq U(d_r^S)$.

Question: How much efficiency is lost by game? Ratio $\vartheta = \frac{U(d_r^G)}{U(d_r^S)}$?

First estimation:



U_r concave & strictly increasing:

$\frac{1}{2}U_r(d_r) \cdot d_r \leq \int_0^{d_r} U_r(z)\,\mathrm{d}z \leq U_r(d_r) \cdot d_r$

$\rightsquigarrow \frac{1}{2}U_r(d_r) \leq U_r^*(d_r) \leq U_r(d_r)$ for all d_r

$\rightsquigarrow \frac{1}{2}U(d_r^S) \leq U^*(d_r^S) \leq U^*(d_r^G) \leq U(d_r^G)$

$\rightsquigarrow \vartheta \geq \frac{1}{2}$

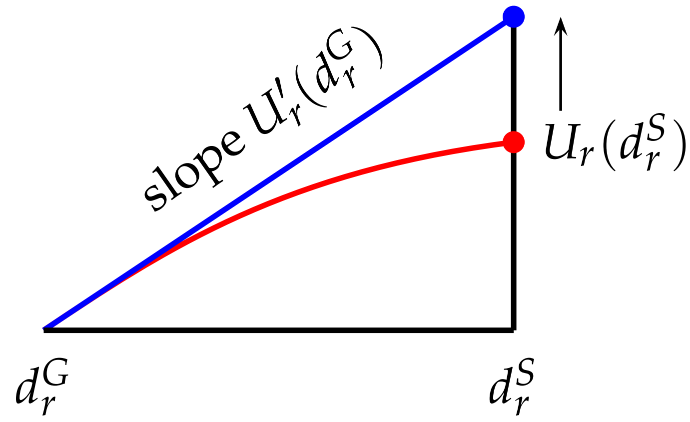
No more than 50% efficiency loss.

Theorem (Johari-Tsitsiklis): At least two users and $U_r(0) \geq 0$ for each user r. Then $\vartheta \geq \frac{3}{4}$ and this bound is tight.

Proof sketch:

- Worst case for *linear* functions U_r.
- Assume $U_r(d_r) = \alpha_r d_r$ with $\alpha_r > 0$.

Scaling & relabeling: w.l.o.g.
$\alpha_1 = \max_r \alpha_r = 1$ and $C = 1$.



- Then: $\mathbf{d}^S = (1, 0, \ldots, 0)$ and $U(\mathbf{d}^S) = 1$.
- We can assume $d_r^G > 0$ for all r.
- Stationary conditions for NE: $U'_r(d_r^G) \cdot (1 - d_r^G) = \lambda$ for all $r \rightsquigarrow$ $\alpha_r \cdot (1 - d_r^G) = 1 - d_1^G$.
- Minimization of $\vartheta = d_1^G + \sum_{r=2}^R \alpha_r d_r^G$ in two stages: first, fix d_1^G and optimize over $d_r^G, r = 2, \ldots, R$ (as function of d_1^G). Then optimize d_1^G.

(Anarchy 1) Minimize $d_1^G + \sum_{r=2}^{R} \frac{(1-d_1^G)d_r^G}{1-d_r^G}$ subject to $\sum_{r=2}^{R} d_r^G = 1 - d_1^G$ and $0 \leq d_r^G \leq d_1^G$.

- For $d_1^G < \frac{1}{R}$ feasible region empty;

for $d_1^G \geq \frac{1}{R}$ non-empty, convex, compact region & objective function strictly convex and continuous: there is unique optimal solution to (Anarchy 1) as function of d_1^G.

- By symmetry: $d_2^G = \ldots = d_R^G$ and $d_r^G = \frac{1-d_1^G}{R-1}$.

(Anarchy 2) Minimize $d_1^G + (1 - d_1^G)^2 \cdot \left(1 - \frac{1-d_1^G}{R-1}\right)^{-1}$ subject to $\frac{1}{R} \leq d_1^G \leq 1$.

- Objective function decreasing in R; worst case in the limit $R \to \infty$:

minimize $d_1^G + (1 - d_1^G)^2$ subject to $0 \leq d_1^G \leq 1$.

Solution $d_1^G = \frac{1}{2} \rightsquigarrow \vartheta \to \frac{3}{4}$ for $R \to \infty$ and bound is tight. □

Interpretation: Worst case for

- Single link with capacity 1;
- Utilities $U_1(d_1) = d_1$ and $U_r(d_r) \approx \frac{d_r}{2}$ otherwise.

For $R \to \infty$ at NE: user 1 receives rate $\frac{1}{2}$ and remaining users uniformly split rate $\frac{1}{2}$.

Remark: Similar to $\frac{4}{3}$-bound observed by Roughgarden-Tardos for selfish routing / congestion games with *linear* latency functions.

But relationship between the two games remains open.

Note: price of anarchy in selfish routing may be unbounded for *non-linear* latency functions while last theorem still holds.

Part 4

Extension to General Networks

The Network Model

- A model of (finite) networks with alternative routing.
- Network = set of paths; paths consist of links with individual capacities.
- Each path belongs to unique user (by duplicating paths: no loss of generality); each user may have several paths.

Formally: For $R, J, P \in \mathbb{N}$:

- Users $r = 1, \ldots, R$.
- Links $j = 1, \ldots, J$ with capacities $C_j > 0$; $\mathbf{C} = (C_1, \ldots, C_J)$.
- Paths $p = 1, \ldots, P$. Path = set of links. Mapping paths $\rightarrow$ users.
- Path-link incidence matrix A: $A_{jp} = 1$ iff link j belongs to path p.
- Path-user incidence matrix H: $H_{rp} = 1$ iff path p belongs to user r.

Generalized Optimization Problem

(System) Maximize $U(\mathbf{d})$ subject to $A\mathbf{y} \leq \mathbf{C}$, $H\mathbf{y} = \mathbf{d}$ and $\mathbf{y} \geq 0$.

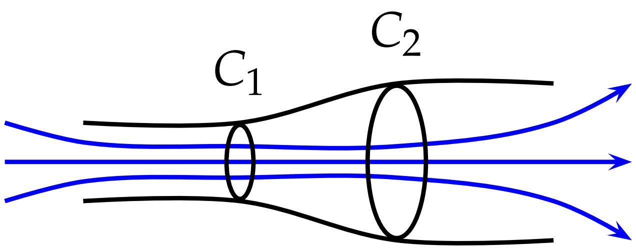
- $\mathbf{y} = (y_1, \ldots, y_p)$: y_p is rate allocated to *path* p.
- $A\mathbf{y} \leq \mathbf{C}$: for each link j, sum of rates of paths containing j is $\leq C_j$.
- $H\mathbf{y} = \mathbf{d}$: rate d_r allocated to user r is sum of rates of user's paths.

Pricing scheme (price anticipating users):

1. Each user submits a bid w_{jr} for each link: $\mathbf{w}_r = (w_{1r}, \ldots, w_{Jr})$.
2. Manager determines price for each link and link rate x_{jr} as before (with $x_{jr} = 0$ if $w_{jr} = 0$).
3. Path rates y_p and rate allocation $d_r(\mathbf{w})$ are determined by solution to *max-flow problem* with constraints x_{jr} (linear program): this guarantees that user r is allocated maximum possible rate d_r.

User's payoff: $Q_r(\mathbf{w}_r; \mathbf{w}_{-r}) = U_r(d_r(\mathbf{w})) - \sum_j w_{jr}$.

Example 2: Bottleneck Problem



- Two links with $C_1 < C_2$.
- Each user has exactly one path consisting of both links.

Scheme: Bids w_{1r} & w_{2r} $\rightsquigarrow$ in proportion: link rates x_{1r} & x_{2r} $\rightsquigarrow$ max-flow problem: allocated rate $d_r = \min\{x_{1r}, x_{2r}\}$.

Suppose $\mathbf{w}$ is NE.

Show: $\sum_r w_{1r} > 0$ and $\sum_r w_{2r} > 0$ (otherwise at least one user can profitably deviate).

But: $\sum_r x_{1r} = C_1 < C_2 = \sum_r x_{2r}$ $\rightsquigarrow$ exists r with $x_{1r} < x_{2r}$ $\rightsquigarrow$ $d_r = x_{1r}$. Thus bidding $(w_{1r}, w_{2r} - \delta)$ for δ small enough reduces payment without altering rate allocation. Contradiction.

Thus: NE may not exist, even if $R > 1$.

Solution: Extended Game

Problem: Link 2 is *not congested*: it will never be fully utilized.

Total bidding $\sum_r w_{2r}$ to link positive $\rightsquigarrow$ one user is always overpaying. Total bidding is zero $\rightsquigarrow$ all users are allocated zero rate.

Discontinuity $x_{jr} = 0$ if $w_{jr} = 0 \rightsquigarrow$ discontinuity of Q_r.

Possible solution: Users not only submit bids, but also *rate requests*. Requests are only taken into account if total bidding to a link is zero.

Formally: Users submit bid w_{jr} and rate request φ_{jr} for each link j.

$$x_{jr} = \begin{cases} \frac{w_{jr}}{\sum_s w_{js}} \cdot C_j & \text{, if } \sum_s w_{js} > 0 \\ \varphi_{jr} & \text{, if } \sum_s w_{js} = 0 \text{ and } \sum_s \varphi_{js} \leq C_j \\ 0 & \text{, if } \sum_s w_{js} = 0 \text{ and } \sum_s \varphi_{js} > C_j. \end{cases}$$

Last case: precise definition not essential; any preset deterministic rule splitting the capacity is possible.

Examples Revisited

Extended game addresses both problems:

- Link not in sufficient demand (Example 1)
- Link not congested (Example 2)

Example 1: Single user submits bid $w = 0$ and rate request $\varphi = C$.

Example 2:

- $(w_{11}, \ldots, w_{1R})$ NE for *single link* game over link 1.
- x_{1r} corresponding rate allocation.
- φ_{1r} arbitrary.
- $w_{2r} = 0$ and $\varphi_{2r} = x_{1r}$.

Since $\sum_r x_{1r} = C_1 < C_2$ the requests will be granted.

Check: $(\mathbf{w}, \boldsymbol{\varphi})$ is NE.

Results for Extended Game (Johari, Tsitsiklis, 2004)

The results for single link extend to general networks:

- Existence of NE, even if $R = 1$.
- NE of the original game $\mathcal{G}$ correspond naturally to NE of the extended game $\mathcal{G}^*$: If $\mathbf{w}$ bidding for $\mathcal{G}$ resulting in link rate $x_{jr} \rightsquigarrow$ set $\varphi_{jr} = x_{jr}$. Then:
  - Each user receives same payoff in either game.
  - $\mathbf{w}$ is NE in $\mathcal{G} \rightsquigarrow (\mathbf{w}, \boldsymbol{\varphi})$ is NE in $\mathcal{G}^*$.
- Partial converse: $(\mathbf{w}, \boldsymbol{\varphi})$ is NE in $\mathcal{G}^*$ with $\sum_r w_{jr} > 0$ for all $j \rightsquigarrow \mathbf{w}$ is NE in $\mathcal{G}$.
- Price of anarchy: Efficiency loss of no more than 25%.
- In fact the restrictions to utility functions U_r can be relaxed: *non-decreasing*, *continuous* and *concave* functions are sufficient.

Proof Ideas

Existence of NE:

- Consider *perturbed* version of $\mathcal{G}$ with additional user submitting always bids $\varepsilon > 0$ to each link (prevents discontinuity at 0).
- Max-flow: d_r is solution to linear program, thus as function of $\mathbf{x}_r$ increasing, continuous & concave $\rightsquigarrow$ payoff concave & continuous.
- Perturbed game can be viewed as *concave R-person game* $\rightsquigarrow$ Rosen's existence theorem guarantees NE.
- Take limit $\varepsilon \to 0$. This works because NE and allocated link rates for perturbed game lie in compact sets.

Relaxation for utility functions:

Determining price of anarchy without differentiability: make use of theory of *supergradients*.

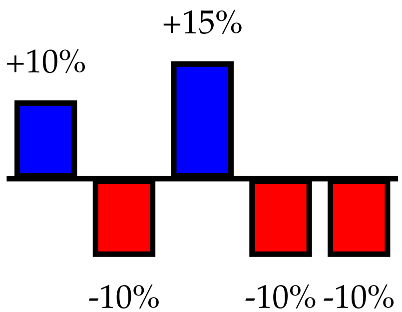
Open question: Uniqueness of NE.

Excursion: Fairness Criteria and Utility Functions

Several fairness criteria correspond to particular families of utilities:

Definition: A feasible rate allocation (RA) $\mathbf{d}$ is called *proportionally fair* if for any other feasible RA $\overline{\mathbf{d}}$:

$$\gamma = \sum_r \frac{\overline{d}_r - d_r}{d_r} \leq 0.$$



Meaning: Any change in the rates must have negative average change.

Lemma: Proportional fair RA is unique and corresponds to (System) with $U_r(d_r) = \ln(d_r)$ for all r.

Proof sketch: In this case: $\nabla U(\mathbf{d})^t \cdot (\overline{\mathbf{d}} - \mathbf{d}) = \gamma$. Hence $\mathbf{d}$ is proportional fair iff $\mathbf{d}$ maximizes U iff $\mathbf{d}$ solves (System).

Since each U_r is strictly concave the solution is unique.

Max-Min Fairness

Definition: A feasible RA $\mathbf{d}$ is called *max-min fair* if for any other feasible RA $\overline{\mathbf{d}}$: if $\overline{d}_r > d_r$ for some r, then there exists s with $d_s \leq d_r$ and $\overline{d}_s < d_s$.

- An increase of any rate must be at the cost of a decrease of some already smaller rate (while maintaining feasibility).
- Max-min fairness gives absolute priority to smaller rates. If a max-min fair RA exists then it is unique.
- There is a relationship between max-min fair RA and bottlenecks.
- The algorithm of progressive filling can be used to obtain the max-min fair RA.

Theorem: Suppose that $d_r < 1$ for all r (adjust capacities). Then max-min fair RA is the limit $\alpha \to \infty$ of solutions to (System) with $U_r^{(\alpha)}(d_r) = -(-\ln(d_r))^\alpha$, $\alpha \geq 1$ for all r.

Extension for Price Taking Users (Sketch)

Pricing scheme: User r submits single bid $w_r \geq 0$ for entire network. Manager determines rate allocation $d_r \geq 0$ proportional to w_r using "total price" $\lambda_r > 0$.

User's payoff: $P_r(w_r; \lambda_r) = U_r\left(\frac{w_r}{\lambda_r}\right) - w_r$.

Again, (System) can be decomposed into $R+1$ subsidiary problems, where the price vector $(\lambda_1, \ldots, \lambda_R)$ mediates between the problems:

Theorem (Kelly): There exists vectors $\mathbf{w} \geq 0$, $\mathbf{d} \geq 0$, and $\boldsymbol{\lambda} > 0$:

- Each w_r maximizes $P_r(w_r; \lambda_r)$;
- $\mathbf{d}$ solves (Manager) Maximize $\sum_{r:w_r>0} w_r \cdot \ln(d_r)$
  subject to $A\mathbf{y} \leq \mathbf{C}$, $H\mathbf{y} = \mathbf{d}$ and $\mathbf{y} \geq 0$;
- $w_r = \lambda_r \cdot d_r$ for all r;
- $\mathbf{d}$ is also solution to (System).

- The former definition of a *competitive equilibrium* for a single link matches with this statement.
- (Manager) corresponds to a *proportionally fair* allocation of rates.
- The proof uses primal and dual formulation of (Manager). The Lagrangian of the problem leads to

$$\lambda_{r(p)} \begin{cases} = \sum_{j \in p} \mu_j & \text{, if } d_r > 0 \\ \leq \sum_{j \in p} \mu_j & \text{, if } d_r = 0, \end{cases}$$

where $r(p)$ denotes the user possessing path p and μ_j is the shadow price of link j. Hence, the aggregate price for every user's path is the same.

Conclusion

Network resource allocation: users submit bids; manager allocates rates in proportion.

Concave, strictly increasing & continuously differentiable utility functions.

Price taking: exists competitive equilibrium, i.e. the social optimum can be decomposed into subsidiary problems (one for each user & one for manager), where a Lagrange multiplier mediates.

Price anticipating: if each user submits bid & rate request for each link, then there is a Nash equilibrium. In fact, non-decreasing, continuous and concave functions suffice.

Price of anarchy: Selfish behavior of users versus social optimum: Efficiency loss of no more than 25%.

Fairness criteria: Proportional fairness is associated to logarithmic utility functions. Max-min fairness is associated to limit of social optimum for appropriate utility functions.

Open questions: Uniqueness of NE. Relation to congestion games / traffic routing.

Thank you for your attention.