Atomic Selfish Routing in Networks: A Survey

GI-Dagstuhl Seminar: Game Theoretic Analyses of the Internet

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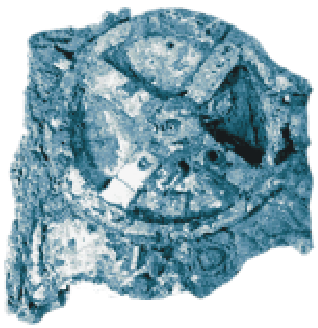

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Related Bibliography

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Finite Strategic Games

- **Strategic game:** A tuple $\langle N, (\mathcal{P}^i)_{i\in N}, (U^i)_{i\in N} \rangle$ where
  - N is the set of players,
  - $\forall i \in N,\ \mathcal{P}^i$ is the set of allowable actions for player i,
  - $\mathcal{P} \equiv \times_{i\in N} \mathcal{P}^i$ is the actions space of the game,
  - $\forall i \in N,\ U^i : \times_{i\in N} \mathcal{P}^i \mapsto \mathbb{R}$ is player i's utility function.
- **Pure strategies:** Each player i chooses an action from $\mathcal{P}^i$ with certainty.
- **Mixed strategies:** Each player i chooses an action **independently of other players** according to a probability distribution over $\mathcal{P}^i$.
- **Strategies profile:** A collection of (mixed in general) strategies for all the players. All players adopt pure strategies $\Rightarrow$ Configuration
- $N, \mathcal{P}^i$ are considered to be **finite** here.

Rational Behavior of Players

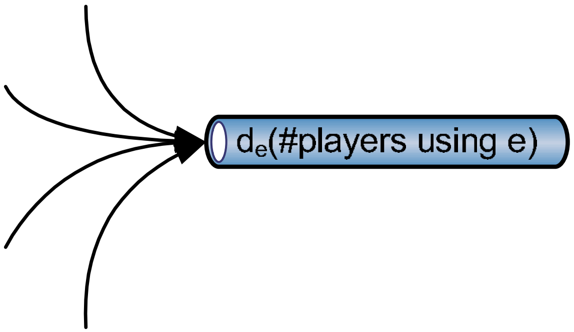
- Each player chooses a strategy that *maximizes* its own private utility U^i, given the other players' strategies (**utility implies preference**).
- **Nash Equilibrium:** A strategies profile s.t. no player has the incentive to change *unilaterally* its own strategy [Nash, 1951] $\Rightarrow$ ALWAYS EXISTS IN FINITE GAMES!
- Some typical challenges:
  - A Pure NE may not exist $\Rightarrow$ DECIDABILITY
  - A mixed NE always exists $\Rightarrow$ COMPUTABILITY ($\in P$?)
  - Many NE may exist $\Rightarrow$ WHICH IS BEST?

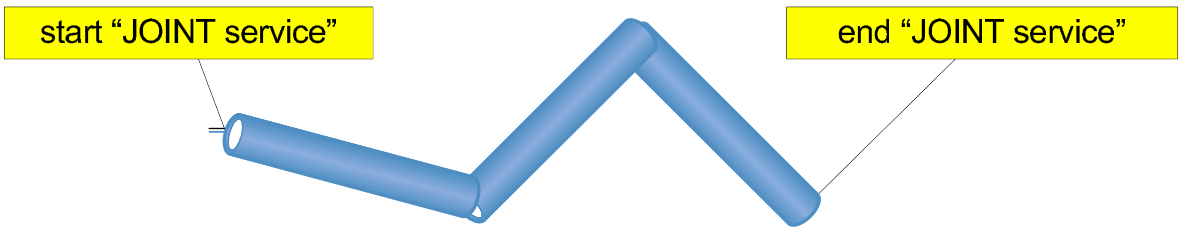
In This Talk

- Unweighted congestion games
  - Equivalence with exact potential games
  - Construction of PNE in polynomial time
  - Player-specific payoffs
- Weighted congestion games
  - Non-existence of PNE
  - Construction of PNE in pseudopolynomial time
  - Price of anarchy
- Recap & Open Problems

Unweighted Congestion Games

[Rosenthal 1973, Monderer & Shapley 1996]





- A set E of shared resources.
- A set N of **non-cooperative** players with **identical** service demands ($\forall i \in N,\ w_i = 1$).
- $\forall i \in N,\ \mathcal{P}^i \subseteq 2^E \setminus \emptyset$ is the set of allowable actions for player i (action = a non-empty collection of resources).
- Each resource $e \in E$ has a **non-decreasing** delay function $d_e : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$, depending only on the **cumulative congestion** (ie, #players using the same resource).

Some Notation

- $\varpi \in \mathcal{P} \equiv \times_{i\in N}\mathcal{P}^i$: A configuration (ie, pure strategies profile) of all the players.
- $\mathbf{p} \in \times_{i\in N}\Delta(\mathcal{P}^i)$: A mixed strategies profile.
- $\varpi^{-i} \in \mathcal{P}^{-i} \equiv \times_{j\neq i}\mathcal{P}^j$: A configuration of all players except for i.
- $\mathbf{p}^{-i} \in \times_{j\neq i}\Delta(\mathcal{P}^j)$: A mixed profile of all players except for i.
- $\mathbf{p}^{-i} \oplus q^i$: the new profile with player i choosing the (mixed in general) strategy $q^i \in \Delta(\mathcal{P}^i)$.
- $P(\mathbf{p},\varpi) = \prod_{i\in N} p^i(\varpi^i)$: the probability of configuration $\varpi \in \mathcal{P}$ occurring, when the players adopt the mixed profile $\mathbf{p} \in \times_{i\in N}\Delta(\mathcal{P}^i)$.

Player's Cost $=-$ Player's Utility

- For any *pure strategies profile* $\varpi \in \times_{i\in N}\mathcal{P}^i$, the selfish cost of player i taking action $\varpi^i \in \mathcal{P}^i$ is:

$$\boxed{\lambda^i(\varpi) = \lambda_{\varpi^i}(\varpi) = \sum_{e\in\varpi^i} d_e(x_e(\varpi))}$$

where, $\Lambda_e(\varpi) \equiv \left\{i \in N : e \in \varpi^i\right\}$ is the set of players using resource e according to ϖ, and $x_e(\varpi) \equiv |\Lambda_e(\varpi)|$ is the **#players** using resource e wrt ϖ.

- For any *mixed strategies profile* $\mathbf{p}$, the selfish cost of player i taking action $\alpha \in \mathcal{P}^i$ is the expectation of the corresponding random variable [von Neumann & Morgenstern 1944]:

$$\boxed{\lambda^i_\alpha(\mathbf{p}) = \sum_{\varpi^{-i}\in\mathcal{P}^{-i}} P(\mathbf{p}^{-i},\varpi^{-i}) \cdot \sum_{e\in\alpha} d_e\left(x_e(\varpi^{-i}\oplus\alpha)\right)}$$

Price of Anarchy

[Koutsoupias & Papadimitriou, 1999]

- **Social Cost**: A *global measure* of system's performance as a function of the strategies profiles. **HERE**:

$$\mathrm{SC}(\mathbf{p}) = \sum_{\varpi \in \mathcal{P}} P(\mathbf{p}, \varpi) \cdot \max_{i \in N}\{\lambda_{\varpi^i}(\varpi)\}$$

(ie, the **expected maximum cost** paid by the players wrt $\mathbf{p}$).

- **Social Optimum of the game**: The *optimum social cost* over all possible configurations of the players

$$\mathrm{OPT} = \min_{\varpi \in \mathcal{P}}\{\max_{i \in N}[\lambda_{\varpi^i}(\varpi)]\}$$

- **Price of Anarchy**: A measure of the game's quality

$$\mathcal{R} = \max_{\mathbf{p} \text{ is a NE}} \left\{ \frac{\mathrm{SC}(\mathbf{p})}{\mathrm{OPT}} \right\}$$

Categories of Congestion Games

A congestion game is...

- symmetric, if all players are *indistinguishable* (ie, have the same action set and use the same utility function which is symmetric on other players' actions) – otherwise we call it asymmetric.
- a (multi-commodity) network congestion game, if for each player i, its allowable actions are all (s_i, t_i)-paths in a graph whose edges are the shared resources of the game.
- a single-commodity network congestion game if all allowable actions of the players are (s, t)-paths in the graph of resources.

Potential Games

[Rosenthal 1973, Monderer & Shapley 1996]

- $\Gamma = \langle N, (\mathcal{P}^i)_{i\in N},\ (U^i : \mathcal{P} \mapsto \mathbb{R})_{i\in N}\rangle$: A strategic game, where $\mathcal{P} \equiv \times_{i=1}\mathcal{P}^i$ is the **actions space**.

- **Neighboring Configurations:** Assume that $\forall\varpi \in \mathcal{P},\ \forall i \in N,\ \forall\alpha \in \mathcal{P}^i \setminus \{\varpi^i\}$,

$$\boxed{\varpi^{-i} \oplus \alpha \equiv (\varpi^1, \varpi^2, \ldots, \varpi^{i-1}, \alpha, \varpi^{i+1}, \ldots, \varpi^n)}$$

Then ϖ and $\varpi^{-i} \oplus \alpha$ are neighboring configurations. Player i is the unique deviator for these two configurations.

Potential Games (contd.)

- For a strategic game Γ, a function $\Phi : \mathcal{P} \mapsto \mathbb{R}$ is called:
  - an ordinal potential iff $\forall i \in N, \forall \varpi \in \mathcal{P}, \forall \alpha \in \mathcal{P}^i$,

    $$\boxed{U^i(\varpi) - U^i(\varpi^{-i} \oplus \alpha) > 0 \Leftrightarrow \Phi(\varpi) - \Phi(\varpi^{-i} \oplus \alpha) > 0}$$

  - a **b-**potential for some $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n_{>0}$, iff $\forall i \in N, \forall \varpi \in \mathcal{P}, \forall \alpha \in \mathcal{P}^i$,

    $$\boxed{U^i(\varpi) - U^i(\varpi^{-i} \oplus \alpha) = b_i \cdot \left(\Phi(\varpi) - \Phi(\varpi^{-i} \oplus \alpha)\right)}$$

  - an exact potential, iff it is a **1**-potential.

Properties of Potential Games

[Monderer & Shapley 1996]:

- A path in $\mathcal{P}$ is a sequence of configurations $\gamma = \langle \varpi(0), \varpi(1), \ldots \rangle$ such that $\forall k \geq 1$ there exists a unique player i_k (the **unique deviator**) such that $\varpi(k) = \varpi(k-1)^{-i_k} \oplus \pi_{i_k}$ for some action $\pi_{i_k} \in \mathcal{P}^{i_k} \setminus \{(\varpi(k-1))_{i_k}\}$.

- γ is an improvement path if $\forall k \geq 1,\ U^{i_k}(\varpi(k)) > U^{i_k}(\varpi(k-1))$ where i_k is the unique deviator at step k.

- The Nash Dynamics graph of the game is the union of all improvement paths in the game.

Definition 1 *A game has the Finite Improvement Property (FIP) if every improvement path has finite length.*

Properties of Potential Games (contd.)

Theorem 1 [Monderer & Shapley 1996]
Every finite ordinal potential game has the Finite Improvement Property.

Corollary 2 *Every finite ordinal potential game has at least one Pure Nash Equilibrium.*

Unweighted Congestion vs. Exact Potential Games

Definition 3 *Two strategic games* $\Gamma = \langle N, (\mathcal{P}^i)_{i\in N}, (U^i)_{i\in N}\rangle$ *and* $\hat{\Gamma} = \langle N, (\hat{\mathcal{P}}^i)_{i\in N}, (\hat{U}^i)_{i\in N}\rangle$ *are* *isomorphic* *iff there is a* ***bijective map*** $g : \mathcal{P} \mapsto \hat{\mathcal{P}}$ *s.t.* $\forall \varpi \in \mathcal{P}, \forall i \in N,\ U^i(\varpi) = \hat{U}^i(g(\varpi))$.

Theorem 2 ([Monderer & Shapley 1996]**)**

(a) *Every unweighted congestion game is an exact potential game.*

(b) *Every (finite) exact potential game is isomorphic to an unweighted congestion game.*

Unweighted Congestion $\mapsto$ Exact Potential

- $C = \langle N, E, (\mathcal{P}^i)_{i\in N}, (d_e)_{e\in E}\rangle$: An arbitrary unweighted congestion game.
- $\boxed{\Phi(\varpi) = \sum_{e\in\cup_{i\in N}\varpi^i} \sum_{k=1}^{x_e(\varpi)} d_e(k)}$ (Rosenthal's potential)
- $\varpi \in \mathcal{P}$, $\hat{\varpi} = \varpi^{-i} \oplus \alpha$: Two arbitrary neighboring configurations.
- $\forall e \in [E \setminus (\varpi^i \cup \alpha)] \cup [\varpi^i \bigcap \alpha]$, resource e has the **same load** wrt ϖ and $\hat{\varpi}$.
- $\forall e \in \varpi^i \setminus \alpha$, $x_e(\hat{\varpi}) = x_e(\varpi) - 1$ and $\forall e \in \alpha \setminus \varpi^i$, $x_e(\hat{\varpi}) = x_e(\varpi) + 1$.

$$\begin{aligned}\Phi(\hat{\varpi}) - \Phi(\varpi) &= \sum_{e\in\alpha\setminus\varpi^i} d_e(x_e(\varpi)+1) - \sum_{e\in\varpi^i\setminus\alpha} d_e(x_e(\varpi)) \\ &= \sum_{e\in\alpha} d_e(x_e(\hat{\varpi})) - \sum_{e\in\varpi^i} d_e(x_e(\varpi)) = \lambda^i(\hat{\varpi}) - \lambda^i(\varpi)\end{aligned}$$

Exact Potential $\mapsto$ Unweighted Congestion (I)

- $\Gamma = \langle N, (Y^i \equiv [m_i])_{i\in N}, (U^i)_{i\in N}, \Phi : Y \mapsto \mathbb{R}\rangle$: An exact potential game.
- $C = \langle N, E, (\mathcal{P}^i)_{i\in N}, (d_e)_{e\in E}\rangle$: The isomorphic congestion game.
- The set of resources: $E = E_1 \cup E_2$

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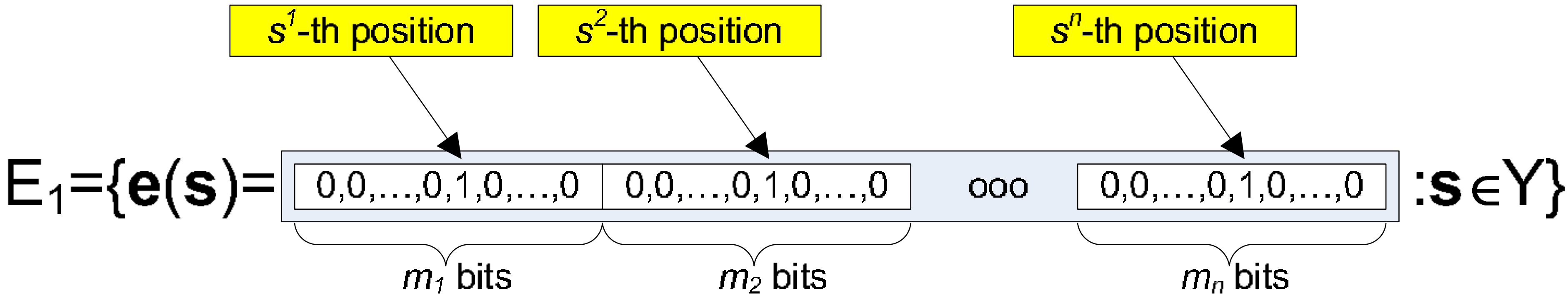

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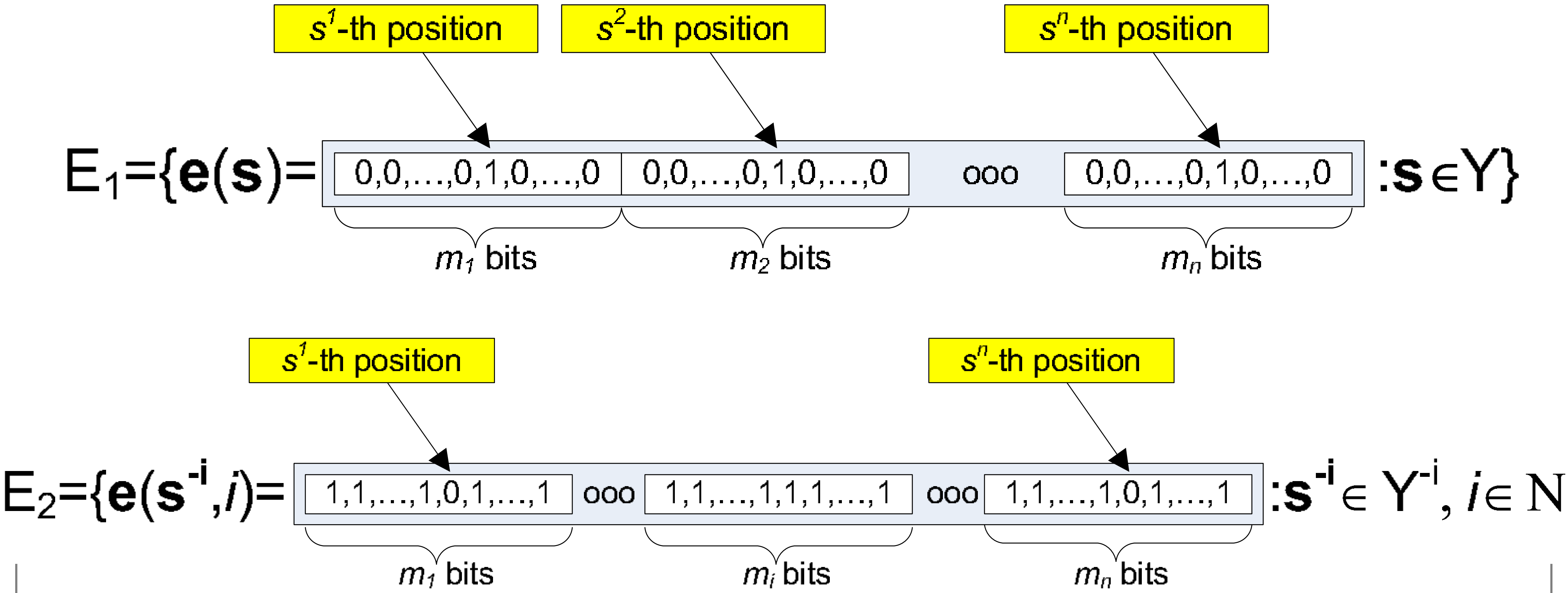

Exact Potential $\mapsto$ Unweighted Congestion (II)

- Players' action sets: $\forall i \in N,\ \mathcal{P}^i = \{\pi^i_1, \ldots, \pi^i_{m_i}\}$ where, $\forall j \in Y^i = [m_i], \pi^i_j = \{\mathbf{e} \in E : e^i_j = 1\}$

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- Bijective map: $\forall \mathbf{s} = (s^1, \ldots, s^n) \in Y$, $\boxed{\mathbf{s} \leftrightarrow \varpi(\mathbf{s}) = (\pi^1_{s^1}, \ldots, \pi^n_{s^n})}$

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- **Some crucial remarks:** $\forall \mathbf{s} \in Y$ and its image $\varpi(\mathbf{s})$ in C,

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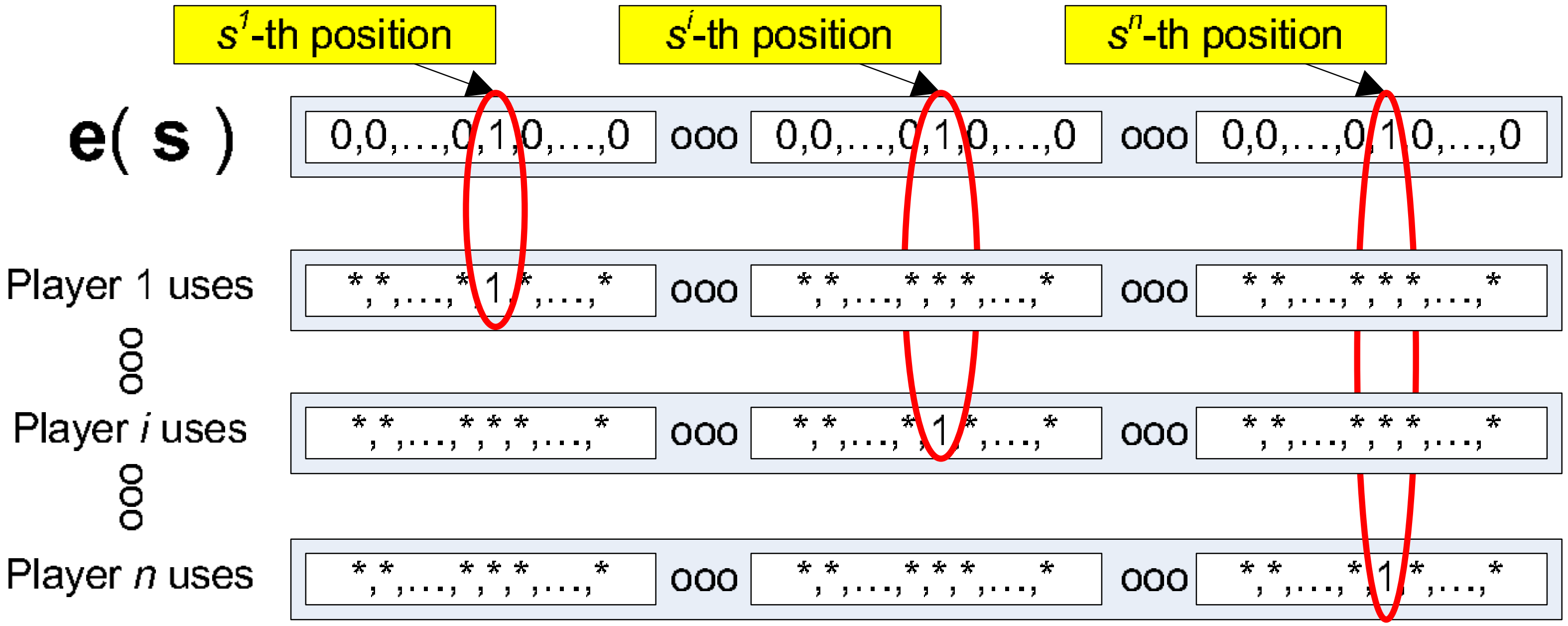
  - **Only** $\mathbf{e}(\mathbf{s})$ in E_1 is used by **all** the players in C.



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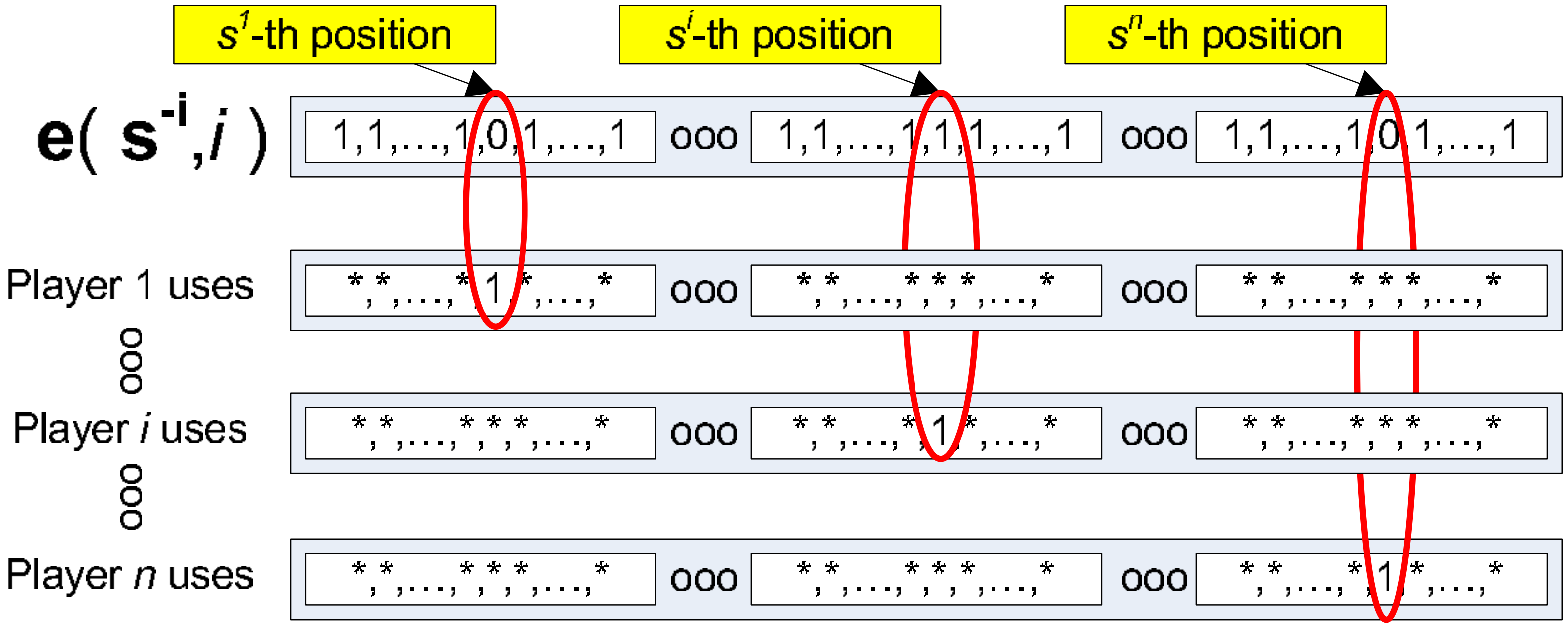
  - **Only** $\mathbf{e}(\mathbf{s})$ in E_1 is used by **all** the players in C.
  - $\forall i \in N$, **only** $\mathbf{e}(\mathbf{s}^{-\mathbf{i}}, i)$ in E_2 is **exclusively used** by player i in C.



Exact Potential $\mapsto$ Unweighted Congestion (III)

- **1st Observation** The players' costs in C: $\forall i \in N, \forall \varpi = \varpi(\mathbf{s}) \in \mathcal{P}$,

$$\begin{aligned}\lambda^i(\varpi(\mathbf{s})) &= \sum_{e \in \varpi^i \bigcap (E \setminus \cup_{j \neq i} \varpi^j)} d_e(1) + \sum_{e \in \cup_{k \neq i} \varpi^i \bigcap \varpi^k \bigcap (E \setminus \cup_{j \neq i,k} \varpi^j)} d_e(2) + \\ &+ \cdots + \sum_{e \in \bigcap_{k \in N} \varpi^k} d_e(n)\end{aligned}$$

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- **2nd Observation** $\forall \mathbf{s} \in Y$, $\forall \alpha \in Y^i \setminus \{s^i\}$,

$$U^i(\mathbf{s}) - U^i(\mathbf{s}^{-\mathbf{i}} \oplus \alpha) = \Phi(\mathbf{s}) - \Phi(\mathbf{s}^{-\mathbf{i}} \oplus \alpha) \Leftrightarrow$$

$$\boxed{Q^i(\mathbf{s}^{-\mathbf{i}}) \equiv U^i(\mathbf{s}) - \Phi(\mathbf{s}) = U^i(\mathbf{s}^{-\mathbf{i}} \oplus \alpha) - \Phi(\mathbf{s}^{-\mathbf{i}} \oplus \alpha)}$$

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- The delay functions: $\forall e \in E$

$$d_e(k) = \begin{cases} \Phi(\mathbf{s}), & \text{if } \mathbf{e} = \mathbf{e}(\mathbf{s}) \in E_1 \wedge k = n \\ Q^i(\mathbf{s}^{-\mathbf{i}}), & \text{if } \mathbf{e} = \mathbf{e}(\mathbf{s}^{-\mathbf{i}}, i) \in E_2 \wedge k = 1 \\ 0, & \text{otherwise} \end{cases}$$

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Efficient Construction of PNEs?

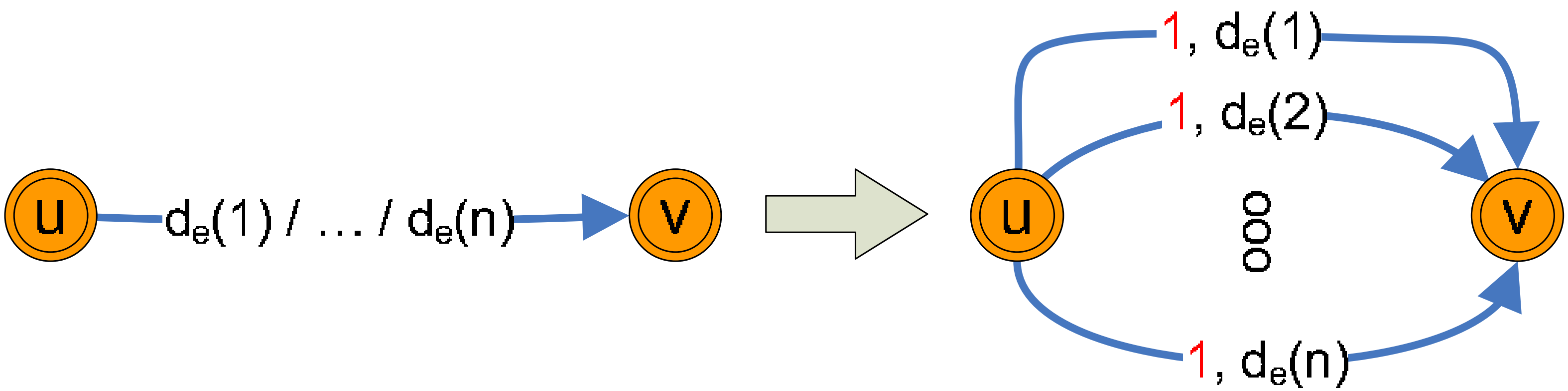
Theorem 3 ([Fabrikant, Papadimitriou, Talwar 2004]**)**

(a) *There is a polynomial time algorithm for constructing a Pure Nash Equilibrium in (**unweighted**) single-commodity network congestion games.*

(b) *It is PLS-complete to construct a Pure Nash Equilibrium even for unweighted* multicommodity network *congestion games.*

An Algorithm for Single-Commodity Networks

- Reduction to MIN-COST FLOW: Transform each resource (arc) e of the initial network into n **capacitated** arcs of a new network with **fixed delays** $d_e(1), \ldots, d_e(n)$:



- Min-cost flow is always **integral**!

- Min-cost flow $\Rightarrow$ **local optimum** for the potential $\Rightarrow$ PNE!

Player Specific Payoffs

[Milchtaich 1996]

- **Main Assumptions:**
  (a) **Parallel links** network
  (b) Players' costs are **non-decreasing** functions of the cumulative congestion.
- **Player Specific Payoffs:** Each player $i \in N$ has his own utility function U_e^i for each parallel link $e \in E$.
- **Results:**

	Unweighted	Weighted
2 strategies	FIP	FIP
2 players	FBRP	FBRP
general case	WA	$\nexists$ PNE

Weighted Congestion Games

- Each player i has a non-negative weight w_i (service demand). The weights are non-identical:

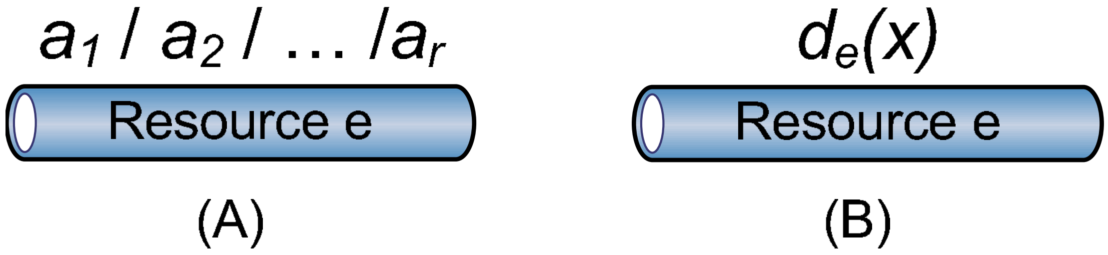
$$\boxed{\mathbf{w} = (w_i)_{i \in N} \in \mathbb{R}_{\geq 0}^{|N|}}$$

- Total load on resource e:

$$\boxed{\theta_e(\varpi) = \sum_{i \in \Lambda_e(\varpi)} w_i}$$

- By definition **asymmetric** games (even for *single commodity* networks).

Some (more) Notation



Meaning:

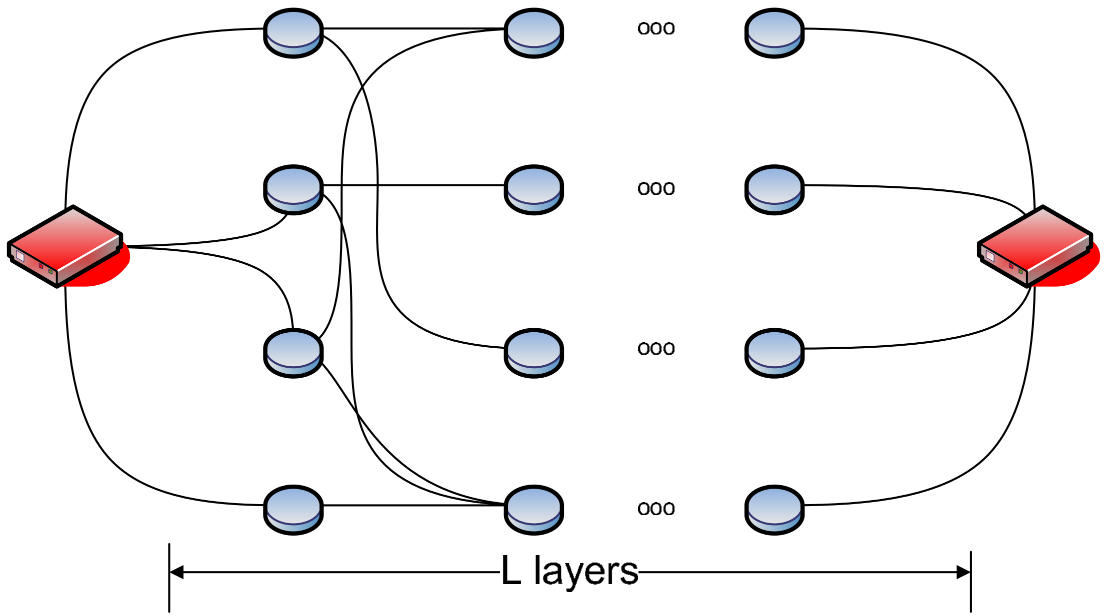
(A) r possible values of total load may appear in resource $e \in E$. For the k^{th} smallest load, the delay of e is a_k.

(B) A continuous function $d_e(x)$ determines the delay of resource e as a function of its load.

Resource Delay Functions:

- **In general:** Non-decreasing functions of loads.
- **Special cases:** Linear delays and two-wise linear delays (ie, maximum of two linear functions).

The Family of L-Layered Networks



- All players want to route traffic from a unique source s to a unique destination t (single-commodity network).
- The nodes are partitioned in $L-1$ groups and lie on (s,t)-paths.
- Edges (representing shared resources) can only exist between nodes of **consecutive** groups of nodes.
- Each (s,t)-path in the network has length exactly L.

$\exists$ PNEs in Weighted Congestion Games?

What we know:

Theorem [Rosenthal 1973, Monderer & Shapley 1996]**:** *Any (**unweighted**) congestion game has at least one Pure Nash Equilibrium.*

∃ Weighted Congestion Games with no PNE

Theorem 4 ([Fotakis,Kontogiannis,Spirakis 2004]**)** *Even a weighted* 3-layered network *game with* 2-wise linear delays *may have no PNE.*

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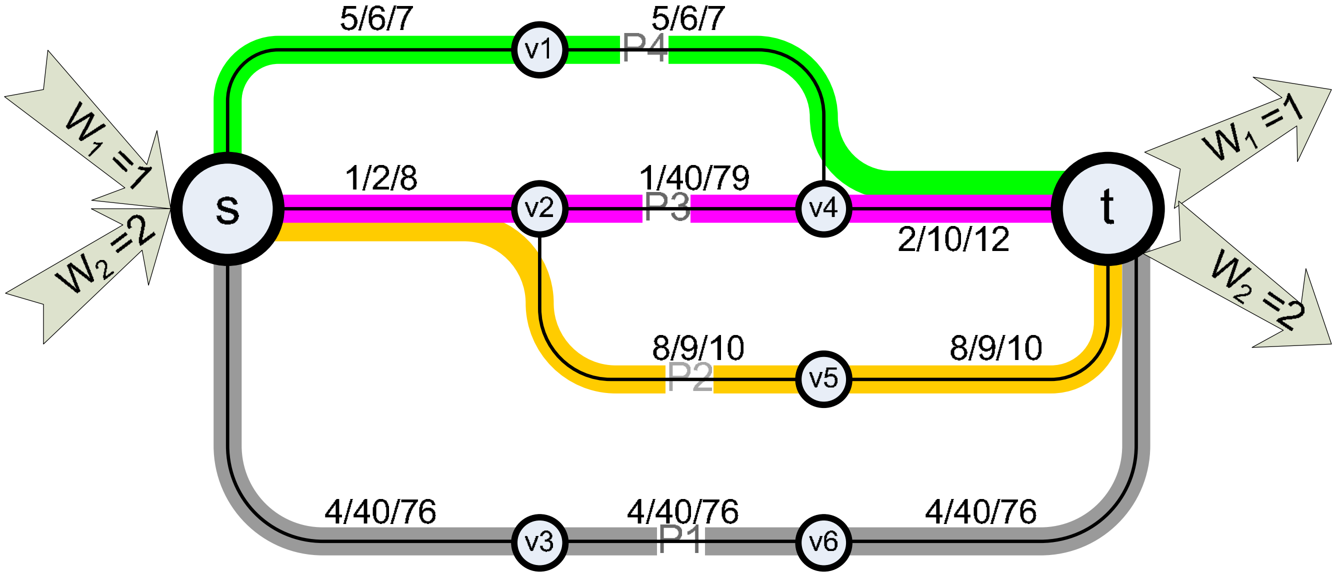
Proof:

- $\exists$ 4-cycle $\langle (P3, P2), (P3, P4), (P1, P4), (P1, P2), (P3, P2) \rangle$ in the **Best Reply Dynamics graph** of the game.



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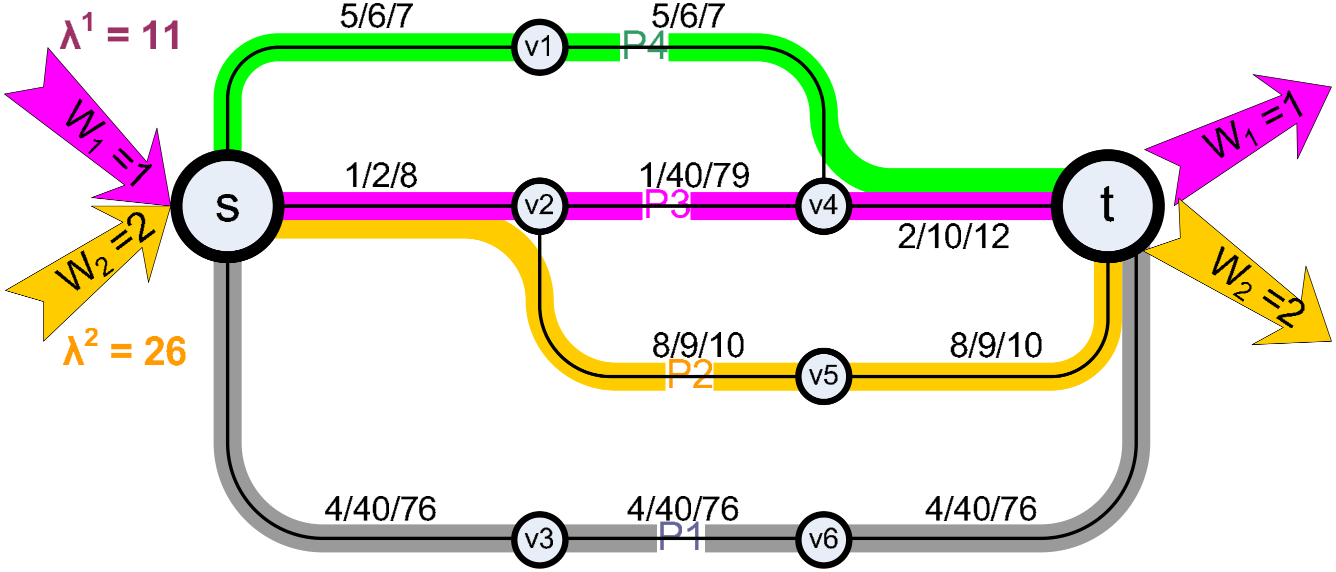

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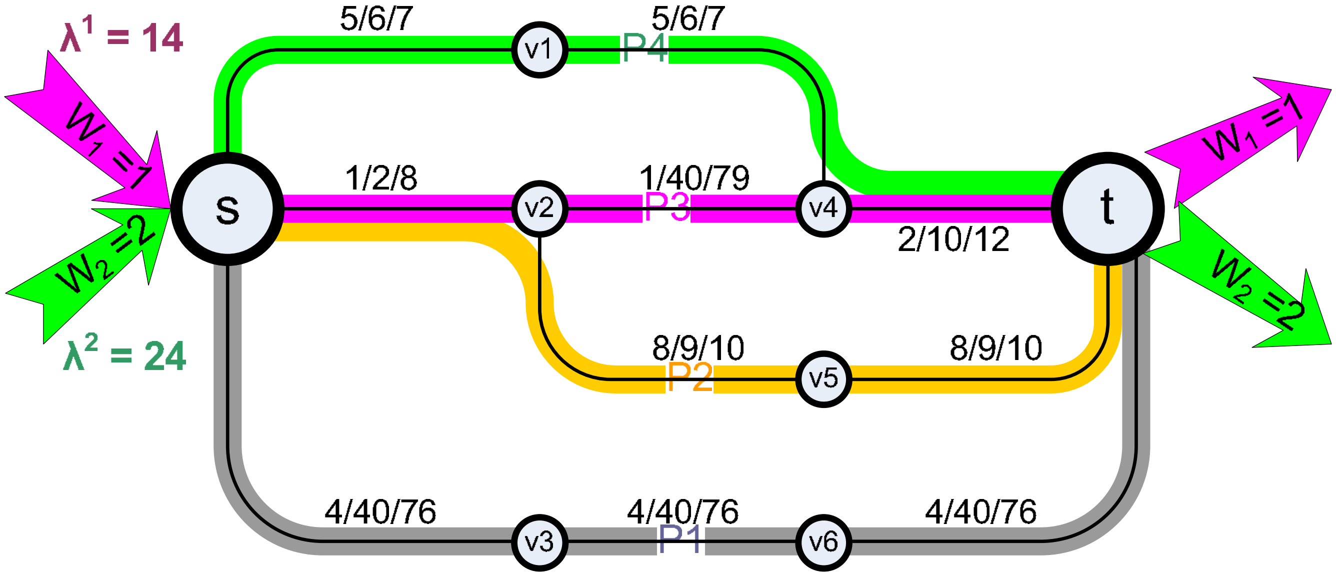

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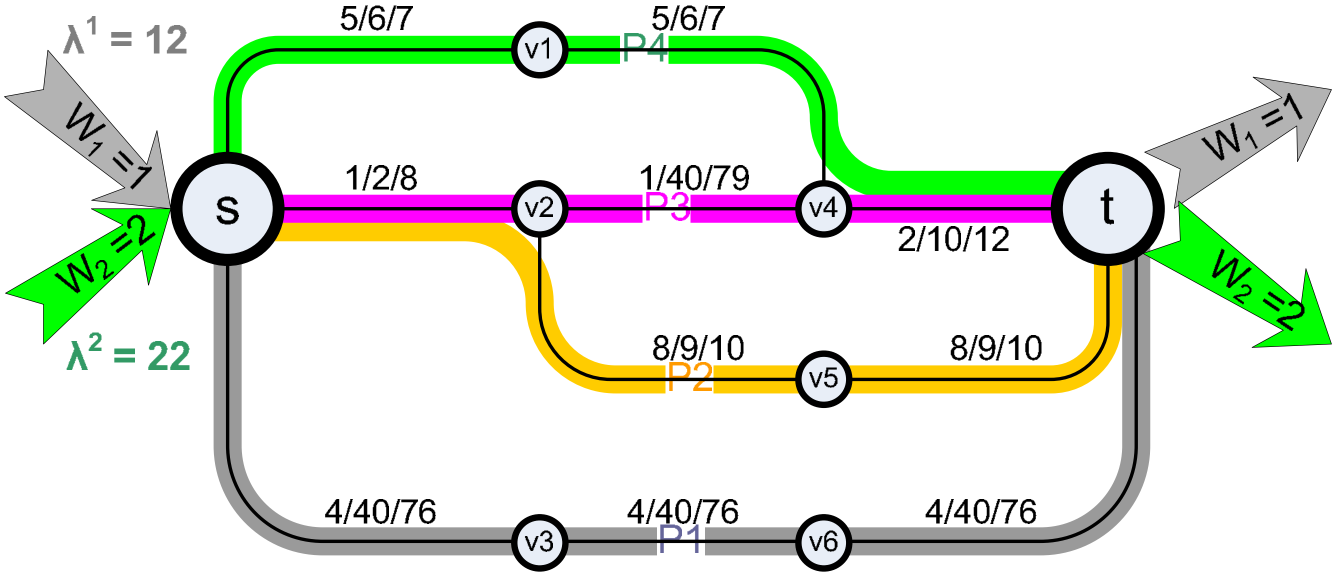

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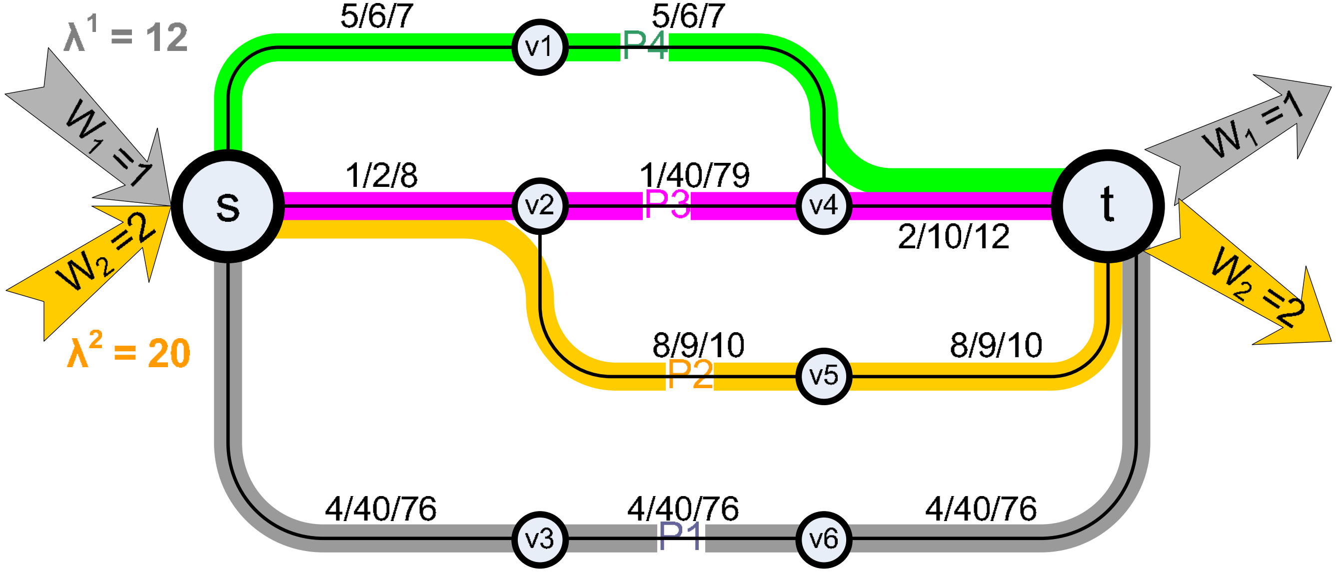

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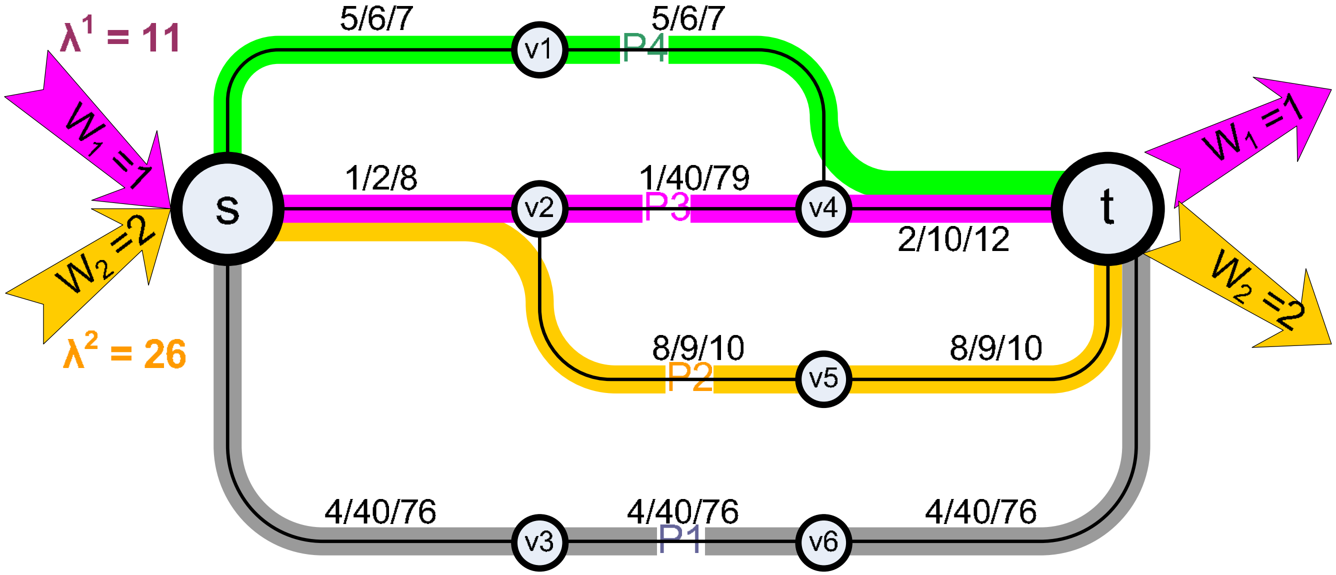

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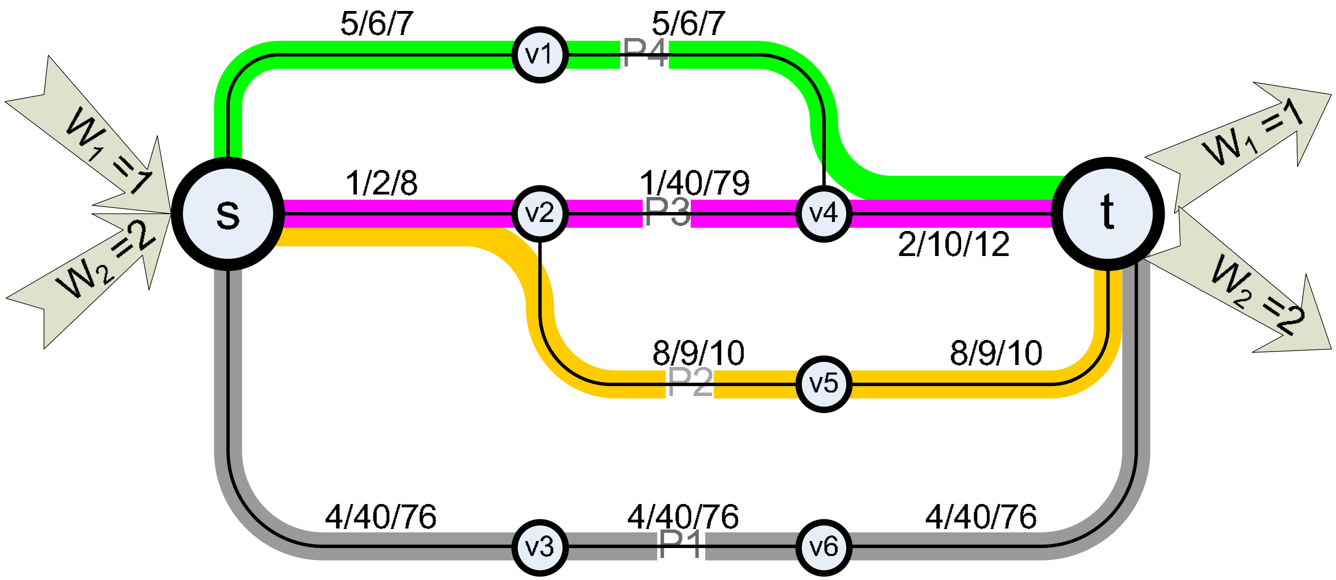

- Any configuration out of this cycle is either **one or two best-reply moves away** from some of its configurations.

∃ Potentials for Weighted Congestion Games?

What we know so far:

- [Rosenthal 1973, Monderer & Shapley 1996] Every (**unweighted**) congestion game admits an *exact potential*, so long as the resource delays are non-decreasing functions of the loads.
- Even **single-commodity** networks with **2-wise linear** resource delays may not admit any kind of potential.
- **Question:** What if we assume **linear resource delays**?

$\nexists$ Exact Potential for Weighted Congestion Games

Theorem 5 ([Fotakis,Kontogiannis,Spirakis 2004]**)** *Even a* single commodity network *congestion game with resource delays* equal *to the loads may not have an exact potential.*

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Proof:

- Cycle = a sequence of configurations $\gamma = \langle \varpi(0), \ldots, \varpi(r) = \varpi(0) \rangle$, where $\forall k,\ i_k$ is the unique player in which $\varpi(k)$ and $\varpi(k-1)$ differ.
- For any cycle γ, $I(\gamma) \equiv \sum_{k=1}^{r} \left[\lambda^{i_k}(\varpi(k)) - \lambda^{i_k}(\varpi(k-1)) \right]$.
- [Monderer & Shapley 1996]: A game admits an *exact potential* iff any 4-cycle γ has $I(\gamma) = 0$.
- The 4-cycle $\gamma = (\varpi,\ \varpi^{-1} \oplus \pi_1,\ \varpi^{-1,2} \oplus \{\pi_1, \pi_2\},\ \varpi^{-2} \oplus \pi_2,\ \varpi)$ has $I(\gamma) = (w_1 - w_2) \cdot \text{NETWORK CONSTANT}$ (typically non-zero).

But ∃ Weighted Potential...

Theorem 6 ([Fotakis,Kontogiannis,Spirakis 2004]**)** *For any weighted* L-layered network congestion game *with resource delays* equal *to their loads, at least one PNE exists and can be constructed in time* $\frac{1}{2}|E|W_{tot}^2$.

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Proof:

- $\Phi(\varpi) = \sum_{e\in E}[\theta_e(\varpi)]^2$ is a $\left(\frac{1}{2w_i}\right)_{i\in N}$-potential for the game $\Rightarrow \exists$ PNE.
- Wlog assume that players have *integer* weights.
- Each arc in the *Nash Dynamics Graph* decreases the potential by at least at least $2w_{\min} \geq 2$.

$\Rightarrow$ Any *improvement path* in the Nash Dynamics graph has length $\leq \frac{1}{2}|E|W_{\text{tot}}^2 \Rightarrow$ pseudopolynomial construction time for PNE.

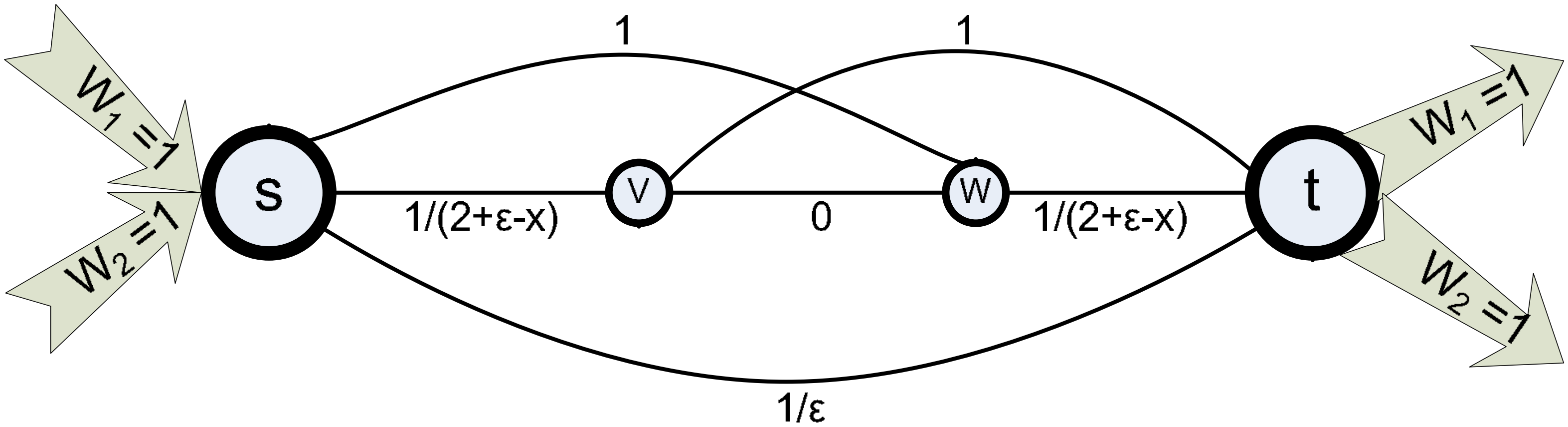
A Recent Improvement

For any weighted congestion game on an arbitrary *multi-commodity network* with *linear* resource delays (ie, $\forall e \in E, d_e(x) = a_e \cdot x + b_e$), there is a PNE that can be constructed in **pseudo-polynomial time**.

[Fotakis, Kontogiannis, Spirakis 2004b]

What about the Price of Anarchy?

Example of [Roughgarden & Tardos 2000] for atomic flows (can easily be transformed into a 3-layered network congestion game):



- Identical users.
- Constant and M/M/1-like resource delays.
- **OPT**:(svt, swt) **NASH**: (st, svwt)
- $\mathcal{R} = \frac{1+\varepsilon}{(2+\varepsilon)\cdot\varepsilon}$

An Example with Linear Delays

Theorem 7 ([Fotakis,Kontogiannis,Spirakis 2004]**)** *The Price of Anarchy can be unbounded, even in unweighted layered network congestion games with linear resource delays.*

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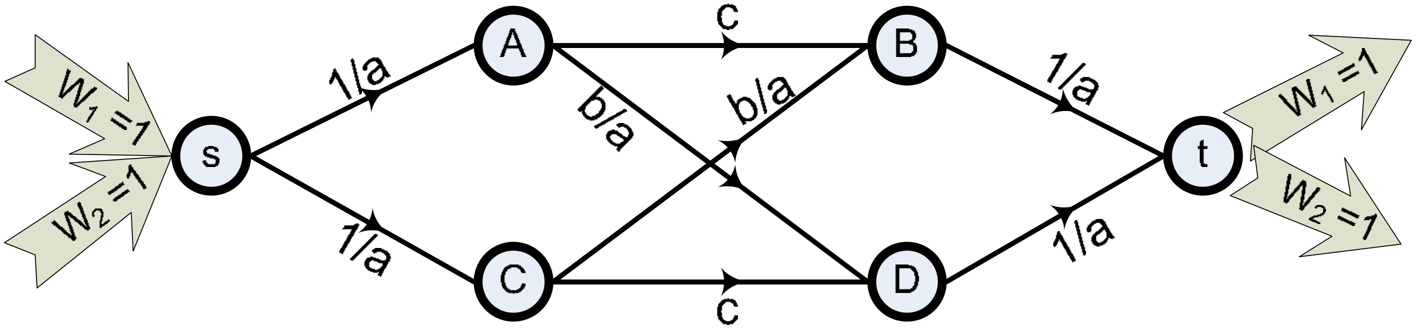

- *Linear* resource delays.
- $a \gg b \gg 1 \geq c \geq 0$.
- **OPT**:(sABt,sCDt)
- **NASH**: (sADt, sCBt)
- $\mathcal{R} = \frac{2+b}{2+c}$

What Remains?

What is the price of anarchy for...

(I) ...*weighted* congestion games on *layered* networks with resource delays *identical* to the loads?

(II) ...*unweighted* congestion games on *general single-commodity* networks with resource delays *proportional* to the loads?

(III) ...*weighted* congestion games on arbitrary *single commodity* networks with *proportional* resource delays?

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⇒ Resolved in this paper

(II) ...*unweighted* congestion games on *general single-commodity* networks with resource delays *proportional* to the loads?
⇒ A recent improvement

(III) ...*weighted* congestion games on arbitrary *single commodity* networks with *proportional* resource delays?
⇒ Remains OPEN

Price of Anarchy in Layered Networks

- $G = (V, E)$: An arbitrary L-layered network.
- P: The set of all (simple) paths in G connecting the **unique** source-destination pair $(s,t) \in V^2$.
- N: The set of $n = |N|$ players.
- Players have distinct weights determined by a function $w : N \mapsto \mathbb{R}_{>0}$.
- Resource delays are **identical** to their loads: $\forall e \in E,\ d_e(x) = x$.

Flows and Mixed Strategies Profiles

- **Feasible Flow**: A function $\rho : P \mapsto \mathbb{R}_{\geq 0}$ s.t. $\sum_{\pi \in P} \rho(\pi) = \sum_{i \in N} w(i)$ (all players' demands are met)

- **Unsplittable Flow**: Each player's demand is routed via a **unique** s-t path.

- **Splittable Flow**: The demand of each player can be split over several s-t paths.

- Mapping mixed strategies profiles to **feasible splittable** flows: $\forall \mathbf{p} \in \times_{i \in N} \Delta(\mathcal{P}^i)$,

$$\boxed{\forall \pi \in P,\ \rho_{\mathbf{p}}(\pi) = \sum_{i \in N} w(i) \cdot p^i(\pi)}$$

NOTE: Each **pure strategies profile** maps to an *unsplittable* flow.

Flow Latencies vs Expected Delays

- The expected delay at a resource $e \in E$ wrt the profile $\mathbf{p}$ equals its **expected load**:

$$\theta_e(\mathbf{p}) \equiv \sum_{i \in N} w(i) \cdot \sum_{\pi \ni e} p^i(\pi) = \sum_{\pi \ni e} \rho_{\mathbf{p}}(\pi) \equiv \theta_e(\rho_{\mathbf{p}})$$

(ie, the **expected delay** of a resource e wrt to a mixed profile = total flow along e = the **latency** of e).

- The expected delay along a path $\pi \in P$ wrt the profile $\mathbf{p}$ is

$$\theta_\pi(\mathbf{p}) = \sum_{e \in \pi} \theta_e(\mathbf{p}) = \sum_{\pi' \in P} |\pi \cap \pi'| \rho_{\mathbf{p}}(\pi') \equiv \theta_\pi(\rho_{\mathbf{p}})$$

(ie, the **expected delay** along $\pi \in P$ is equal to the **cumulative latency** along this path caused by the corresponding flow).

Measures of Flows

- Maximum Latency of a flow $\rho = \rho_{\mathbf{p}}$:

$$L(\rho) \equiv \max_{\pi:\rho(\pi)>0} \{\theta_\pi(\rho)\} = \max_{\pi:\exists i\in N, p^i(\pi)>0} \{\theta_\pi(\mathbf{p})\} \equiv L(\mathbf{p})$$

(ie, the **maximum latency** caused by the flow $\rho = \rho_{\mathbf{p}}$ is equal to the **maximum expected delay** paid by the players wrt to the mixed strategies profile $\mathbf{p}$).

- Total Latency of a flow $\rho = \rho_{\mathbf{p}}$:

$$C(\rho) \equiv \sum_{\pi\in P} \rho(\pi)\theta_\pi(\rho) = \sum_{e\in E} \theta_e^2(\mathbf{p}) \equiv C(\mathbf{p})$$

(no direct connection to the associated mixed strategies profile).

- $\theta^{\min}(\rho) \equiv \min_{\pi\in P}\{\theta_\pi(\rho)\}$ is the min path latency in the network (ie, min expected delay of an additional **negligible** demand traveling from s to t).

Anarchy of Weighted Layered Network Games

Theorem 8 ([Fotakis,Kontogiannis,Spirakis 2004]**)** *The Price of Anarchy in any L-layered network with resource delays* ***equal*** *to the loads, is at most* $8e\left(\frac{\log m}{\log\log m}+1\right)$, *where* $m = |E|$.

NOTE: The **parallel links** network (which is a **1-layered network**) is essentially the worst layered network in this case!

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MAIN STEPS OF A USEFUL MACHINERY:

(I) **Bounding Step**: Bound the *maximum expected delay* of any NE as a function of the *social optimum*.

NOTE: This is also an upper bound on the price of anarchy of PNE.

(II) **Statistical Conflict**: Prove that when moving from *maximum expected delay* to *expected maximum delay* (ie, the social cost), the blow-up is $O\left(\frac{\log m}{\log\log m}\right)$.

(I.a) Connecting the two Flow Measures

Lemma 4 *For resource delays* ***equal*** *to their loads, there is a* unique splittable flow ρ_f^* *that minimizes* ***both*** $L(\rho)$ *and* $C(\rho)$.

Corollary 5 *The optimum splittable flow* ρ_f^* *defines a* symmetric NE *for the players:*

$$\boxed{\forall i \in N,\ \forall \pi \in P,\ p^i(\pi) = \frac{\rho_f^*(\pi)}{W_{tot}}}$$

(I.b) Some Quadratic Programming

Lemma 6 *Any* $\mathbf{p}$ *at NE has* $L(\mathbf{p}) \leq 3 \cdot L(\rho^*) = 3 \cdot OPT$*, where* ρ^* *is the* optimal unsplittable flow.

(I.b) Some Quadratic Programming

Lemma 6 *Any* $\mathbf{p}$ *at NE has* $L(\mathbf{p}) \leq 3 \cdot L(\rho^*) = 3 \cdot OPT$, *where* ρ^* *is the* optimal unsplittable flow.

- $\forall \pi, \pi' \in P^2,\ Q[\pi, \pi'] = |\pi \cap \pi'|$.
- The matrix Q is **symmetric** and **positive semi-definite**.
- For any feasible flow ρ, $C(\rho) = \rho^T Q \rho$, while $(Q\rho)_{\pi \in P} = \theta_\pi(\rho)$.

(QP) $$\min\{\rho^T Q \rho :\ \mathbf{1}^T \rho \geq W_{\text{tot}};\ \rho \geq \mathbf{0}\}$$

(DP) $$\max\{zW_{\text{tot}} - \rho^T Q \rho :\ 2Q\rho \geq \mathbf{1}z;\ z \geq 0\}$$

- For any f.s. ρ of (QP), $(\rho, 2\theta^{min}(\rho))$ is a f.s. of (DP).
- (QP)-(DP): Optimal solutions of the **same value** (by strong duality).

(I.b) Some Quadratic Programming (contd.)

- ρ^*_f: The optimal solution of (QP) $\Rightarrow$ $(\rho^*_f, 2\theta^{min}(\rho^*_f))$ is a f.s. of (DP) with objective $C(\rho^*_f)$ $\Rightarrow$ Dual optimum solution.
- For any f.s. ρ of (QP), $2\theta^{min}(\rho)W_{\text{tot}} - C(\rho) \le C(\rho^*_f)$.
- For any f.s. ρ at **Nash** equilibrium,

$$\begin{aligned} L(\rho) &\le \theta^{min}(\rho) + Lw_{max} \le \frac{C(\rho)+C(\rho^*_f)}{2W_{\text{tot}}} + Lw_{max} \\ &\le \frac{1}{2}[L(\rho)+L(\rho^*_f)] + Lw_{max} \Rightarrow \\ L(\rho) &\le L(\rho^*_f) + 2Lw_{max} \le 3 \cdot L(\rho^*) \end{aligned}$$

(II) Statistical Conflict

Lemma 7 *Let*

- ρ^*: *the optimal* unsplittable *flow wrt the Max Latency objective.*
- $\rho = \rho_{\mathbf{p}}$: *the feasible flow corresponding to a mixed strategies profile* $\mathbf{p}$.

If $L(\mathbf{p}) \leq a \cdot L(\rho^*) = a \cdot OPT$ *for some constant* $a \geq 1$ *then*

$$\boxed{SC(\mathbf{p}) \leq (a+1) \cdot 2e \frac{\log m}{\log\log m} \cdot OPT}$$

Proof Sketch

- $\forall i \in N,\ \forall e \in E,\ X_{e,i} = w(i) \cdot \mathbb{I}_{[i \text{ uses } e]}$ and $\mathbb{E}[X_{e,i}] = w(i) \sum_{\pi \ni e} p^i(\pi)$
- $X_e = \sum_{i \in N} X_{e,i}$: the r.v. for the **actual load** on resource $e \in E$.
- Due to independence of $\{X_{e,i}\}_{i \in N}$ apply Hoeffding + Union:

  $\exists e \in E : \mathbb{P}[X_e \geq \mathrm{ek} \max\{\theta_e(\rho), w_{max}\}] \leq |E| \cdot k^{-\mathrm{ek}}$
- $X_\pi \equiv \sum_{e \in \pi} X_e$: the r.v. for the **actual delay** (ie, cumulative load) on path $\pi \in P$.
- Use the following facts:
  - $\forall e,\ X_e \leq \mathrm{ek}_0 \max\{\theta_e(\rho), w_{max}\} \Rightarrow \forall \pi,\ X_\pi \leq \mathrm{ek}_0(a+1)L(\rho^*)$
  - $SC(\mathbf{p}) = \mathbb{E}\left[\max_{\pi:\rho(\pi)>0}\{X_\pi\}\right] \leq e(a+1)L(\rho^*)[k_0 + 2mk_0^{1-ek_0}]$
  - Set $k_0 = \frac{2\log m}{\log\log m}$

A Recent Complementary Result

The price of anarchy of an arbitrary **single-commodity network** congestion game with players of **unit demands** and resource delays are **proportional** to their loads (ie, $\forall e \in E,\ d_e(x) = a_e \cdot x$), is

$$\boxed{24e\left(\frac{\log m}{\log\log m}+1\right)}$$

[Fotakis, Kontogiannis, Spirakis 2004b]

Recap and Open Problems

Resolved Issues:

- A complete characterization on the existence of PNE and given pseudo-polynomial algorithms (whenever they exist).
- The Price of Anarchy for most cases of weighted congestion games.

Remain Open:

- Price of Anarchy for players with *different demands* and single-commodity networks with *proportional* delays?
- Price of Anarchy in *multicommodity network* congestion games with *linear/proportional/identical* delays?
- In which cases there is a *polynomial time algorithm* for constructiing PNE (or always PLS-Complete)?

Thanks!