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Algorithmic Game Theory

Sample Exam Questions

Problem 1. *Nash Equilibria I* (12 Points)

a) Find all Nash Equilibria of the following 2-player strategic game. Players aim at maximizing the payoff. (6 Points)

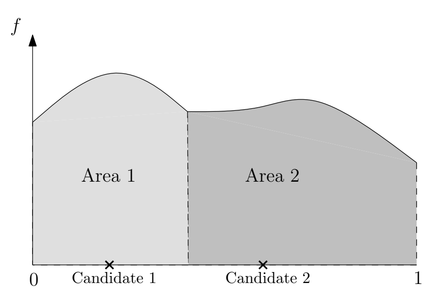
		Colin a	b	c	d
Rose	A	$(0,4)$	$(1,3)$	$(-2,1)$	$(-4,1)$
	B	$(2,0)$	$(1,4)$	$(5,5)$	$(3,2)$

b) Consider the following strategic game with 2 players called *Avoider* and *Guesser*. Fix a constant $k \geq 2$. Both players pick a number in $\{1,\dots,k\}$. If the two numbers are different, the payoff of both players is zero. If the two numbers are equal, Avoider's payoff is -1 and Guesser's payoff is $+1$. Find the unique Nash Equilibrium of this game and prove that this is actually a Nash Equilibrium (we do not ask you to prove that the Nash Equilibrium is unique). (6 Points)

Problem 2. *Nash Equilibria II* (12 Points)

Suppose there are n politicians that choose whether to become a candidate for the next election and, if so, with which political belief. A possible political belief is a number in the unit interval $[0,1]$.

We assume that the population is a continuum, each of whom has a political belief and ready to vote for the candidate with the belief closest to herself. The continuous density function f over the unit interval shall describe how many persons have a certain belief. We assume that $f(x) > 0$ for every belief x. See the example below for two candidates. The area below the curve denotes the number of voters for every candidate.



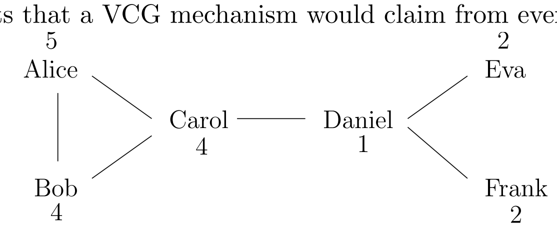
If k candidates have the same belief x, they each receive a fraction $1/k$ of all the votes that x attracts. The winner of the election is the candidate that attracts more votes. Each candidate prefers to win the election than to tie for the first place, prefers to tie for the first place than to stay out of the competition and prefers to stay out of the competition than to enter and lose.

a) Find all pure Nash Equilibria of this game for $n = 2$. Explain why there are no other pure Nash Equilibria than the one(s) you found. You might want to use $x^\star$ as the belief with the property that half of the population has a belief smaller than $x^\star$ and half of the population has a belief bigger than $x^\star$. (5 Points)

b) Prove that for $n = 3$ no pure Nash Equilibrium exists. (7 Points)

Problem 3. *Auctions* (10 Points)
We are running a multi-unit auction for badminton rackets in the town Stutxhöchi, where nobody owns one yet and we are the only supplier. We assume that we give at most one racket to each player and that we are auctioning k rackets. Of course, being the only person to own a badminton racket is no fun; bidders care about which other bidders win rackets as well. Let us suppose that players that live close enough to each other can play badminton together. This can be represented as a graph G, which has an edge between two players if and only if they live close enough to each other to play together. A player i's utility is as follows: i has utility 0 if she does not get a racket and has utility of $v_i \cdot n_i$ if she gets a racket, where $v_i > 0$ is the player's type and n_i is the number of neighbors of i in the graph that get a racket.

a) Consider the graph depicted below, where the number noted by each player corresponds to the reported bid. Suppose that we have two rackets to auction that are granted to Alice and Bob (an optimal allocation). Declare the payments that a VCG mechanism would claim from every player. (4 Points)



b) A clique in a graph is a subset of the nodes that are pairwise adjacent. The *Clique Problem* is the problem of determining whether a graph contains a clique of at least a given size k. This is a decision problem (the output is either "yes" or "no") and it is known to be NP-complete. Show that the Allocation Problem of the aforementioned auction (with general graphs and bids) is NP-hard by reducing the Clique Problem to it. Specifically, go over the following points.

(i) Take an instance of the Clique Problem and describe a construction of an instance of the Allocation Problem with the property that a clique of size k exists in the original graph, if and only if the solution of the Allocation Problem in the new graph has a total revenue of at least a well-chosen value that depends on k. Prove that the reduction is correct. (5 Points)

(ii) Argue briefly that the reduction, i.e., the transformation from the Clique Problem instance to the Allocation Problem instance, can be carried out efficiently (i.e., in polynomial time with respect to the original instance). (1 Points)

Problem 4. *Elections* (10 Points)
Let the unit interval $[0, 1]$ be the set of alternatives of an election. There are n voters, each voter i having a single most preferred alternative $t_i \in [0, 1]$. The player i's preferences of other alternatives are decreasing with the increasing distance from t_i, i.e., player i prefers alternative $x \in [0, 1]$ to alternative $y \in [0, 1]$, if (and only if) $|x - t_i| < |y - t_i|$. Player i is indifferent between x and y, if they are at the same distance from t_i. Thus, the preferences of player i are determined by a single value t_i. The voters vote by casting a number from the interval $[0, 1]$ (which is handled by the election as the most preferred alternative). The outcome of the election (as a result of a social choice function) is a single number in the interval $[0, 1]$.

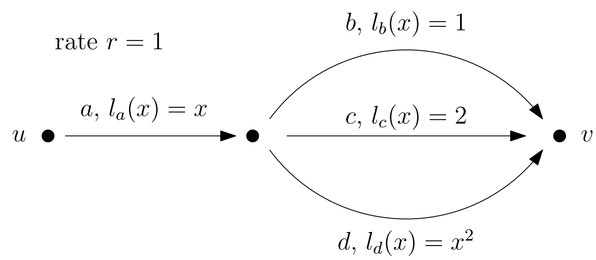
a) Show that the social choice function that returns the average of the casted numbers (claimed types) is not truthful. (3 Points)

b) Devise a truthful social choice function f, which is *onto* and *anonymous*. A social function f is anonymous if $f(s_1, \ldots, s_n) = f(s_{\pi(1)}, \ldots, s_{\pi(n)})$, for every permutation π of the set $\{1, \ldots, n\}$. In other word, f is independent of the name of the players (and thus cannot be a dictatorship). (7 Points)

Problem 5. *Selfish Routing* (6 Points)
Consider the following network on three vertices with links a, b, c, d and the given latencies for every link. The rate of the flow with source in u and sink in v is 1.

a) Find a Nash flow in the depicted network and determine its cost. Justify your answer. (3 Points)

b) Find an optimal flow in the depicted network and determine its cost. Justify your answer. (3 Points)



Problem 6. *Network Formation Games* (10 Points)
During the lecture it has been proved, that for $\alpha = 2$ the complete graph and the star are optimal networks in the Network Formation Game, i.e., they attain the minimum of the social cost function $\alpha \cdot |E| + \sum_{i,j} d(i,j)$ among all networks with n nodes. In this exercise we fix $\alpha = 2$.

a) Determine the social cost for an optimal network with n nodes according to what was said above. (2 Points)

b) Show that the network on vertices $\{1, 2, 3, 4, 5\}$ depicted below is optimal. (2 Points)

c) Prove that a network G is not optimal if G has diameter at least 3, i.e., $d(i,j) \geq 3$ for at least a pair of nodes i, j. (2 Points)

d) Prove that if the diameter of G is at most 2 (that is $d(i,j) \leq 2$ for all i, j) then G is optimal. (4 Points)