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Algorithmic Game Theory FS07

Exercise sheet 3

EXERCISE 3.1:

Let G be a 2-player zero-sum game. Assume there are two pure Nash Equilibria in G constituted by the two respective strategy profiles $s^1 = (s_1^1, s_2^1)$ and $s^2 = (s_1^2, s_2^2)$.

- Prove that the payoffs of the two Nash Equilibria are the same, i.e., show that $u_i(s^1) = u_i(s^2)$, $i = 1, 2$.
- Show that (s_1^1, s_2^2) and (s_1^2, s_2^1) are Nash Equilibria of G as well.

EXERCISE 3.2:

Consider a zero-sum game G between Rose and Colin, where Rose's strategy set is $S_1 = \{1, \dots, n\}$ and Colin's strategy set is $S_2 = \{1, \dots, m\}$. Let $A = (a_{ij})$ be the payoff matrix, i.e., a_{ij} is Rose's payoff if i and j are Rose and Colin's strategy, respectively, and $-a_{ij}$ is Colin's payoff in the same situation.

- Show that $\max_i \min_j a_{ij} \leq \min_j \max_i a_{ij}$ for every matrix A .
- Prove that if $\max_i \min_j a_{ij} = \min_j \max_i a_{ij}$, then the strategy profile (i^*, j^*) , where i^* is the strategy for which $\min_j a_{i^*j} = \max_i \min_j a_{ij}$, and j^* is the strategy for which $\max_i a_{ij^*} = \min_j \max_i a_{ij}$, is a Nash Equilibrium of the game.
- Use the claim in b) to find the Nash Equilibria of the game described by the following matrix.

$$\begin{pmatrix} 4 & 5 \\ -2 & -3 \\ 0 & 6 \end{pmatrix}$$

EXERCISE 3.3:

Consider a 3-player zero-sum game, i.e., a game with the property that for every strategy profile $s = (s^1, s^2, s^3) \in S_1 \times S_2 \times S_3$ it holds that $u_1(s) + u_2(s) + u_3(s) = 0$.

Show that the problem of finding a Nash Equilibrium in this class of games is *at least as hard* as finding a Nash Equilibrium in general 2-player games.

Hint. Given a 2-player game G , show that a 3-player zero-sum game G' can be constructed with the property that the Nash Equilibria of G can be *efficiently* computed, provided that the Nash Equilibria of G' are known.

EXERCISE 3.4:

In the lecture it was proved that the problem of deciding if a strategic game with 2 players, each with n

strategies, admits more than one Nash Equilibrium is *NP-complete*. To do so, we reduce the *Satisfiability Problem* (SAT) to this problem. Roughly, given an instance of SAT (which is a boolean formula φ in a conjunctive normal form) we construct *efficiently* a game G with the property that G has more than one Nash Equilibrium, if and only if the SAT formula is satisfiable.

We briefly recall the construction. Let L , V and C be the set of *literals*, *variables* and *clauses* of a SAT formula φ , respectively. Let $|V| = n$. The function $v : L \rightarrow V$ gives the variable corresponding to a literal. The 2-player game $G(\varphi)$ is defined as follows: the strategy-space of both players is $S := L \cup V \cup C \cup \{f\}$ and the payoffs are listed below.

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| (A) $u_1(f, f) = u_2(f, f) = 0$; | (H) $u_1(v, l) = u_2(l, v) = 2 - n$, $v \in V$, $l \in L$ with $v(l) = v$; |
| (B) $u_1(z, f) = u_2(f, z) = -2$, $z \in S \setminus \{f\}$; | (I) $u_1(v, l) = u_2(l, v) = 2$, $v \in V$, $l \in L$ with $v(l) \neq v$; |
| (C) $u_1(f, z) = u_2(z, f) = 1$, $z \in S \setminus \{f\}$; | (J) $u_1(l, z) = u_2(z, l) = -2$, $l \in L$, $z \in S \setminus L$; |
| (D) $u_1(c, z) = u_2(z, c) = -2$, $c \in C$, $z \in S \setminus L$; | (K) $u_1(l, \bar{l}) = u_2(\bar{l}, l) = -2$, $l \in L$ |
| (E) $u_1(c, l) = u_2(l, c) = 2$, $c \in C$, $l \in L$ with $l \notin c$; | (L) $u_1(l_1, l_2) = u_2(l_2, l_1) = 1$, $l_1, l_2 \in L$ with $\bar{l}_1 \neq l_2$. |
| (F) $u_1(c, l) = u_2(l, c) = 2 - n$, $c \in C$, $l \in L$ with $l \in c$; | |
| (G) $u_1(v, z) = u_2(z, v) = -2$, $v \in V$, $z \in S \setminus L$; | |

We consider assignments of values to variables given by literals, one for each variable, which simply say which literal evaluates to *true*. Thus, an assignment can be given by (l_1, l_2, \dots, l_n) , where l_i determines the value of variable x_i . For example, if x_i is a variable and literal l_i is \bar{x}_i , the variable x_i is assigned a *false* value.

In this exercise you are asked to prove that the formula φ is satisfiable by an assignment (l_1, \dots, l_n) , if and only if the strategy profile where both players play l_i with probability $1/n$, $i = 1, \dots, n$, is a Nash Equilibrium of $G(\varphi)$.

Deadline. You are to hand in your solutions during the lecture on Thursday, October 25th, 8:15-10:00 in CAB G11.