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Algorithmic Game Theory FS07

Exercise sheet 6

EXERCISE 6.1:
In the lecture, we have seen some desirable properties that a voting system should preferably have. We summarize hereafter the definitions. Let F be a *social welfare function* and L the set of all *preferences* of the players over the candidate set A, and $a, b \in A$ be two arbitrary candidates. $(\succ_1, \dots, \succ_n) \in L^n$ is an arbitrary *preference profile* and $\succ = F(\succ_1, \dots, \succ_n)$.

i) F is said to fulfil the *unanimity* condition if the fact that $a \succ_i b$ for every player i, implies $a \succ b$.

ii) F is said to be *consistent* if given two profiles $(\succ_1, \dots, \succ_n)$ and $(\succ'_1, \dots, \succ'_n)$ and the preferences $\succ = F(\succ_1, \dots, \succ_n)$ and $\succ' = F(\succ'_1, \dots, \succ'_n)$, the following holds:

$$\forall i \big(a \succ_i b \Leftrightarrow a \succ'_i b\big) \Longrightarrow \big(a \succ b \Leftrightarrow a \succ' b\big)$$

iii) F is a *dictatorship* if there is a player i who is a *dictator* in F, i.e., a player who dictates the outcome of the election: for every profile $\succ_1, \dots, \succ_n$ it holds that, $F(\succ_1, \dots, \succ_n) = \succ_i$.

We are now ready for the following questions.

a) If F is a dictatorship, how many dictators can it have?

b) Consider elections in which there are only two candidates a and b, i.e., $|A| = 2$. Thus the preference of every player i is either $a \succ_i b$ or $b \succ_i a$. The *majority* vote between two candidates chooses the candidate that is preferred by the majority of the players. In case of a tie, a is chosen. Show that the majority vote, considered as a social choice function, is truthful, i.e., no player can strategically manipulate this voting system (see later for the definition).

c) Can you devise for the case of b) a social welfare function F, which satisfies unanimity, consistency and which is not a dictatorship?

d) Consider a social choice function $f : L^n \to A$, $|A| \geq 3$, which is not onto (that is, at least a candidate can never be elected). Can you devise a truthful social choice function f?

e) How many different social welfare functions that satisfy unanimity and consistency are there, for the setting with n voters and $|A| \geq 3$?

f) What can you say in e), if $|A| = 2$?

EXERCISE 6.2:
Recall the following two definitions. Thereby L is as before the set of all the possible preferences of the players.

i) A social choice function is *truthful* if it cannot be *strategically manipulated*. A player i can strategically manipulate a social choice function f if for some $\succ_1, \ldots, \succ_n \in L$ and some $\succ_i' \in L$, we have $a' \succ_i a$, where $a = f(\succ_1, \ldots, \succ_n)$ and $a' = f(\succ_1, \ldots, \succ_i', \ldots, \succ_n)$.

ii) A social choice function f is *monotone* if for all preference profiles, $f(\succ_1 \ldots, \succ_i, \ldots, \succ_n) = a \neq a' = f(\succ_1 \ldots, \succ_i', \ldots, \succ_n)$ implies that $a \succ_i a'$ and $a' \succ_i' a$.

Prove that a social choice function is truthful if and only if it is monotone.

EXERCISE 6.3:
The proof of *Arrow's theorem* relies on the lemma called *Pairwise Neutrality*. Let us first state the lemma.

Let F be a social welfare function over A, $|A| \geq 3$, which satisfies unanimity and consistency. Further let $\pi_1 = (\succ_1 \ldots, \succ_n)$ and $\pi_2 = (\succ_1' \ldots, \succ_n')$ be two preference profiles and a, b, c, d (possibly not distinct) candidates such that for every player i, it holds that $a \succ_i b$ if and only if $c \succ_i' d$. Then $a \succ b$ if and only if $c \succ' d$, where $\succ = F(\succ_1, \ldots, \succ_n)$ and $\succ' = F(\succ_1', \ldots, \succ_n')$.

Prove the Pairwise Neutrality Lemma by following the given trace.

a) Argue, that if $a = c$ and $b = d$ the claim follows easily and show that otherwise, without loss of generality, we may assume that $a \succ b$ and $c \neq b$.

b) Create a new profile $\pi^\star = (\succ_1^\star, \ldots, \succ_n^\star)$ as follows: for every i take the preference $\succ_i$ and move c just above a (unless $c = a$) and d just below b (unless $d = b$) and leave the other candidates where they are in the ordering $\succ_i$. Let $\succ^\star = F(\succ_1^\star, \ldots, \succ_n^\star)$.
Show that for every i, $a \succ_i b$ if and only if $a \succ_i^\star b$ and $c \succ_i' d$ if and only if $c \succ_i^\star d$.

c) Show that $c \succ^\star a$ and $b \succ^\star d$.

d) Use the assumptions of the lemma to conclude that $c \succ^\star d$ and thus $c \succ' d$.

Deadline. You are to hand in your solutions during the lecture on Thursday, November 15th, 8:15-10:00 in CAB G11.