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Algorithmic Game Theory FS07

Exercise sheet 9

In the last lecture, we considered *Network Formation Games* with n players. We briefly recall here the main ideas of this class of games.

We represent the players as nodes of a graph G , initially devoid of links. Every node has two competing goals: on one hand each node wants to cheaply communicate to every other node of the network (the cost to do so is the length of the shortest path between the two nodes) and on the other hand the node wants to achieve this goal by buying possibly few costly links. A node i 's strategy is to select a set of vertices of G and *buy* the links (edges) that connect i to them. Every edge costs α and we denote by n_i the number of links bought by node i . The cost that node i incurs is then $\alpha \cdot n_i + \sum_j d(i, j)$, where $d(i, j)$ is the distance between i and j in G , i.e., the length of the shortest path connecting i to j in G . Obviously, a rational node aims at minimizing her cost. Note that if node i buys a link ij , then j can use this link as well. Nonetheless, we say that in this case ij *points to* j (or *is oriented towards* j) to stress the fact that i bought the link. As customary, a *Nash Equilibrium* is a strategy profile, such that no player has an incentive to unilaterally change her mind.

The *social cost* of a network is defined as $\sum_{i \neq j} d(i, j) + \alpha \cdot \sum_i n_i$. An *optimal solution* is a solution that minimizes the social cost for a given α .

EXERCISE 9.1:

In this exercise, we consider *stable networks*, i.e., solutions of the game which are Nash Equilibria.

- a) Prove the following (trivial) claims. Let G be a stable network.
 - (i) If ij is a link bought by i , then j did not buy ij as well.
 - (ii) G is a connected graph.
- b) The complete graph is an optimal solution if $\alpha \leq 2$. Show that nonetheless the complete graph with at least three vertices is not a stable network if $\alpha > 1$ (regardless of the orientation of the links).
- c) Propose a stable network for $\alpha > 2$ which is not a star.

EXERCISE 9.2:

In what follows we want to investigate the importance of how links are oriented.

- a) Let $\alpha \geq 1$. Prove that a star is always stable, regardless of how the links are oriented.
- b) Let $\alpha \geq 1$ and pick a path on four vertices. Show that the stability of this network depends on the orientation of the links.

EXERCISE 9.3:

Prove that if $\alpha > n^2$, then any stable network is a tree. Is the opposite true? Further show that the Price of Anarchy is a constant in this case.

Deadline. You are to hand in your solutions during the lecture on Thursday, December 6th, 8:15-10:00 in CAB G11.