Nowhere zero flow

Definition: A flow on a graph $G = (V, E)$ is a pair $(D, f)$ such that

1. $D$ is an orientation of $G$.

2. $f$ is a function on $E$.

3. $\sum_{u \in N_D^+(v)} f(uv) = \sum_{w \in N_D^-(v)} f(vw)$ for every $v \in V$.

Example: $f \equiv 0$.

Definition: A nowhere zero flow on a graph $G = (V, E)$ is a flow $(D, f)$ such that $f(e) \neq 0$ for every $e \in E$. 
Definition: For a positive integer \( k \), a \( k \)-flow on a graph \( G = (V, E) \) is a flow \((D, f)\) such that \( f : E \to \mathbb{Z} \) and \(-(k - 1) \leq f(e) \leq k - 1\) for every \( e \in E \).

Definition: A flow \((D, f)\) on \( G = (V, E) \) is called positive if \( f(e) > 0 \) for every \( e \in E \).

Definition: A \( k \)-flow which is nowhere zero is called a \( k \)-nowhere zero flow or \( k \)-NZF for brevity.

Proposition: The following conditions are equivalent:

1. \( G \) admits a positive \( k \)-flow.

2. \( G \) admits a \( k \)-NZF.

3. Every orientation of \( G \) admits a \( k \)-NZF.

Corollary: Admiting a \( k \)-NZF is a property of the underlying undirected graph.
Which graphs admit a $k$-NZF?

**Theorem:** A graph $G$ admits a $k$-NZF (for some $k$ that might depend on $G$) iff it is bridgeless.

**Lemma:** The flow along any (directed) cut is zero.

**Proof (of the Lemma and the Theorem):** Exercise.

**Question:** Is there a fixed $k$ such that every bridgeless graph admits a $k$-NZF? We will answer this question later.

**Small values of $k$:**

**Observation:** No graph admits a 1-NZF. A graph admits a 2-NZF iff it is even.

**Proposition:** (Tutte 1949) A cubic graph admits a 3-NZF iff it is bipartite.
Flow coloring duality

**Definition:** A $k$-coloring of a graph $G = (V, E)$ is a mapping $f : V \rightarrow \{0, 1, \ldots, k - 1\}$. A coloring $f$ is proper if $f(u) \neq f(v)$ for every edge $(u, v) \in E$.

**Remark:** A graph $G = (V, E)$ admits a $k$-coloring (for some $k$ that might depend on $G$) iff it is loopless. Moreover, a coloring $f : V \rightarrow \{0, \ldots, k - 1\}$ induces a function $g : E \rightarrow \{0, \ldots, k - 1\}$ by setting $g(u, v) = |f(u) - f(v)|$. Note that $g$ is nowhere zero iff $f$ is proper. Moreover, $g$ vanishes along (directed) cycles.

**Definition:** Let $G = (V, E)$ be a plane graph. The dual of $G$ is a graph $G^* = (V^*, E^*)$, where $V^*$ is the set of faces of $G$, and $(f_1, f_2) \in E^*$ iff they share an edge in $G$. Note that, in some sense, $E^* = E$.

**Observation:** If $e \in E$ is a bridge, then $e^* \in E^*$ is a loop. If $e_1, \ldots, e_k \in E$ are the edges of a circuit of $G$, then $e_1^*, \ldots, e_k^* \in E^*$ are the edges of a cut of $G^*$. This can be (somehow) generalized to non-planar graphs.
Flows and colorings of plane graphs

**Theorem:** (Tutte 1954) A plane bridgeless graph $G = (V, E)$ is $k$-face-colorable (its dual is $k$-colorable) iff it admits a $k$-NZF.

**proof:** Let $c$ be a proper $k$-coloring of the faces of $G$. Let $D$ be an arbitrary orientation of $G$. Define a function $f : E \to \mathbb{Z}$ by setting $f(e) = c(f^e_L) - c(f^e_R)$, where $f^e_L$ is the face to the left of $e$ (w.r.t. to its direction under $D$) and $f^e_R$ is the face to the right of $e$.

Conversely, let $(D, f)$ be a $k$-NZF on $G$. Define a $k$-coloring $c$ of the faces of $G$ as follows. Color the unbounded face with color 0. For any other face $F$, traverse faces (in an arbitrary route) to the outer face. Define $c(F)$ to be the (directed) sum of flows through edges you cross. Add $f(e)$ if $e$ points to your right and subtract otherwise. All calculations are modulo $k$. Note that $c$ is a well-defined proper $k$-coloring.
Group flow

**Definition:** Let $\Gamma$ be an abelian group (say, w.r.t. addition). A $\Gamma$-flow on a graph $G = (V, E)$ is a flow $f : E \to \Gamma$. A $\Gamma$-flow $f$, such that $f(e) \neq 0$ (that is, $f(e)$ is not the identity element of $\Gamma$) for every $e \in E$, is called a $\Gamma$ nowhere zero flow or $\Gamma$-NZF for brevity.

**Theorem:** (Tutte 1954) Let $\Gamma$ be a group of order $k$ and let $G$ be a graph. Then, $G$ admits a $k$-NZF iff $G$ admits a $\Gamma$-NZF.

**Theorem:** (Tait 1878) A simple bridgeless cubic plane graph is 3-edge colorable iff it is 4-face colorable.

**Theorem:** A simple bridgeless cubic graph is 3-edge colorable iff it admits a 4-NZF.

**proof:** Use the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. 
Tutte’s flow conjectures

Conjecture: (3-NZF) Every 4-edge-connected graph admits a 3-NZF (dual to Grötzsch’s Theorem).

Conjecture: (4-NZF) Every bridgeless graph containing no subdivision of the Petersen graph admits a 4-NZF (dual to the 4-color Theorem).

Conjecture: (5-NZF) Every bridgeless graph admits a 5-NZF.

Weaker version: There exists a positive integer $k$ such that every bridgeless graph admits a $k$-NZF.
4-NZF in 4-edge-connected graphs

**Theorem:** Every 4-edge-connected graph admits a 4-NZF.

**Proof:** Let $G = (V, E)$ be a 4-edge-connected graph. Let $T_1, T_2$ be two edge disjoint spanning trees of $G$ (Tutte 1961, Nash-Williams 1961).

For $i = 1, 2$ and for every $e \in E \setminus E(T_i)$, let $C^i_e$ denote the unique circuit in $T_i \cup \{e\}$, and let $f^i_e$ be a flow with values $\pm i$ on the edges of $C^i_e$ and 0 elsewhere.

Let $f_1 = \sum_{e \in E \setminus E(T_1)} f^1_e \mod 4$. Note that $f_1(e) = \pm 1$ for every $e \in E \setminus E(T_1)$.

Let $F = \{e \in E(T_1) : f_1(e) = 0 \mod 4\}$, let $f_2 = \sum_{e \in F} f^2_e \mod 4$, and let $f = f_1 + f_2 \mod 4$.

Note that $f$ is a $\mathbb{Z}_4$-NZF.
Solving the weak 5-NZF Conjecture

**Theorem:** (Kilpatrick 1975, Jaeger 1979) Every bridgeless graph admits an 8-NZF.

**Theorem:** (Seymour 1981) Every bridgeless graph $G = (V, E)$ admits a 6-NZF.

**Proof:** Main idea - find a “good” even subgraph $H$ of $G$. Prove that $G \setminus H$ admits a $\mathbb{Z}_3$-NZF $f$. Let $g$ be a $\mathbb{Z}_2$-NZF on $H$; then $(f, g)$ is a $(\mathbb{Z}_3 \times \mathbb{Z}_2)$-NZF on $G$.

**Construction of a good $H$:** We construct a series of graphs $H^0 \subseteq \ldots \subseteq H^r$, where $r$ is the smallest index $i$ such that $H^i$ spans $G$, recursively - $H^0$ is an arbitrary vertex of $V$, and $H^i = (H^{i-1} \cup H_i) + F_i$ for every $1 \leq i \leq r$. 
**Definition of \( H_i \) and \( F_i \):** Let \( X_i \subseteq V \setminus V(H^{i-1}) \) be minimal such that \( X_i \neq \emptyset \) and \( e(X_i, V \setminus (V(H^{i-1}) \cup X_i)) \leq 1 \). Let \( F_i \) be an arbitrary pair of edges connecting \( X_i \) and \( V(H^{i-1}) \) (if there is only one such edge, then take it). Let \( H_i \) be any even connected subgraph of \( G[X_i] \) that contains the “\( X_i \)-endpoints” of the edges in \( F_i \).

**Observations:**

1. \( X_i \) is well defined (as \( X_i = V \setminus V(H^{i-1}) \) is valid).
2. \( e(X_i, V(H^{i-1})) \geq 1 \) (as \( e(X_i, V \setminus (V(H^{i-1}) \cup X_i)) \leq 1 \) and \( G \) is 2-edge-connected).
3. \( G[X_i] \) is 2-edge-connected or a single vertex (as \( X_i \) is minimal).
4. \( H_i \) is well defined (\( G[X_i] \) is 2-edge-connected, Menger).

Let \( G' = G \setminus H^r = (V, E') \) and let \( H = H^0 \cup \bigcup_{i=1}^r H_i \). Note that \( H \) is even and that \( H^r \) is spanning and connected.
**Definition of a $\mathbb{Z}_3$-flow:**

We define $\mathbb{Z}_3$-flows $f_r, f_{r-1}, \ldots, f_0$ in “reverse” induction, such that $f_i$ is nowhere zero on $E' \cup \bigcup_{j=i+1}^{r} F_j$, for every $0 \leq i \leq r$.

**Induction base:** For every $e \in E'$, let $C_e$ denote an arbitrary circuit in $H^r \cup \{e\}$ that contains $e$, and let $f_e$ be a flow with values $\pm 1$ on the edges of $C_e$ and 0 elsewhere. Let $f_r = \sum_{e \in E'} f_e$.

**Induction step:** $f_r, \ldots, f_i \Rightarrow f_{i-1}$

**Case 1:** $|F_i| = 1$, say $F_i = \{e\}$. Let $f_{i-1} = f_i$. We claim that $f_i(e) \neq 0$. It suffices to prove that $f_i(e') \neq 0$, where $e'$ is the unique edge in $E(X_i, V \setminus (V(H^{i-1}) \cup X_i))$. By the induction hypothesis, it suffices to show that $e' \in E' \cup \bigcup_{j=i+1}^{r} F_j$. Clearly, $e' \notin \bigcup_{j=1}^{i} F_j$ and $e' \notin \bigcup_{k=1}^{i-1} H_k$. Moreover, $e'$ is a bridge of $G[V \setminus V(H^{i-1})]$ and for every $i \leq k \leq r$, $H_k$ is a bridgeless subgraph of $G[V \setminus V(H^{i-1})]$. 
Case 2: $|F_i| = 2$, say $F_i = \{e_1, e_2\}$. Since $H^{i-1}_i$ and $H_i$ are both connected, there exists a cycle $C$ in $(H^{i-1}_i \cup H_i) + F_i$ that contains $e_1$ and $e_2$. If $f_i(e_1) \neq 0$ and $f_i(e_2) \neq 0$, then let $f_{i-1} = f_i$. Otherwise assume wlog that $f_i(e_1) = 0$. Circulate a flow of 1 on $C$ in the direction agreeing with $e_2$; denote this flow by $g$. Let $f_{i-1} = f_i + g$. Note that $f_{i-1}(e_1) \neq 0$, $f_{i-1}(e_2) \neq 0$ and $f_{i-1}(e) = f_i(e)$ for every $e \in E' \cup \bigcup_{j=i+1}^{r} F_j$.

The theorem now follows since $H$ is even.
Cycle double cover

**Definition:** A cycle double cover of a graph $G$ is a collection of cycles of $G$ such that every edge of $G$ appears in exactly two of them.

**Examples:** Eulerian graphs and bridgeless planar graphs.

**Conjecture:** (Szekeres 1973) Every bridgeless graph has a cycle double cover.

**Proposition** A graph $G$ admits a 4-NZF iff it has a CDC that forms three even subgraphs.

*Proof:* If $G$ admits a 4-NZF, then it is the union of two even subgraphs $G_1$, $G_2$. The cycles of $G_1$, $G_2$ and $G_1 \Delta G_2$ yield the desired cover. Let $H_1$, $H_2$ and $H_3$ be even graphs formed by the cover, then $G = H_1 \cup H_2$. Hence, $G$ admits a 4-NZF.