Graphs & Algorithms: Advanced Topics
Planar Separators

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Separators

Definition
Let $G = (V, E)$ be a graph on $n$ vertices, $f : \mathbb{N}_0 \to \mathbb{R}$ a function, $\alpha \in (0, 1)$ a parameter. $S \subseteq V$ is a $(f(n), \alpha)$-separator if
- $|S| \leq f(n)$ and
- all connected components of $G - S$ are of size $\leq \alpha n$.

Example
Every tree contains a $(1, \frac{1}{2})$-separator, which can be found in $O(n)$ time.

Applications
For example, divide-and-conquer algorithms (more details later).
Planar Separators

Theorem (Lipton–Tarjan 1979)

Every planar graph $G$ on $n$ vertices has a $(\sqrt{8n}, \frac{2}{3})$-separator, which can be found in time $O(n^3)$.  

Remarks

- $f(n) = \Theta(\sqrt{n})$ is best possible if we want $(f(n), \alpha)$-separators for all planar graphs, for a constant $\alpha < 1$.
- More generally, one can find $(\Theta(\sqrt{n}), \frac{2}{3})$-separators in graphs with a fixed forbidden minor $H$ (Alon–Seymour–Thomas 1990; algorithm runs in time $O(n^{3/2})$; constants depending on $H$).
- **Sparsity** of the graph (i.e., a linear number of edges) is not enough to guarantee the existence of small separators. E.g., for fixed $r \geq 3$, with high probability, a random $r$-regular graph $G$ is vertex expanding, i.e., any subset $A \subseteq V$, $|A| \leq |V|/2$ has at least $\varepsilon \cdot |A|$ neighbors outside of $A$, for some constant $\varepsilon$ depending on $r$. Thus, such a $G$ has no $(f(n), \alpha)$-separator with $\alpha < 1$ and $f(n) = o(n)$. 
Divide-and-Conquer method

1. **BASE CASE**: Solve the problem on constant-size sets by brute force.

2. Otherwise **DIVIDE**: Find a “very small” vertex set $C$ “fast” such that $G - C$ falls into two “small” pieces $A$ and $B$ with no edges in between.

3. **CONQUER**: Explore all solutions restricted to $C$ (brute force) and solve the corresponding subproblems on $A$ and $B$ recursively. Put together the partial solutions.

Here:
- “Small” means $< \beta n$, where $\beta < 1$ is a constant.
- “very small” means $o(n)$.
- Outcome: Algorithm with subexponential running time $2^{\text{very small}}$.
Planar Independent Sets

**MAXIMUM (PLANAR) INDEPENDENT SET**

**Input:** (Planar) graph $G$

**Output:** Independent set $X \subseteq V$ with maximum cardinality, that is, $|X| = \alpha(G)$.

**Theorem**

The **MAXIMUM PLANAR INDEPENDENT SET problem** is NP-hard.

**Theorem**

The **MAXIMUM PLANAR INDEPENDENT SET problem** can be solved in time $2^{O(\sqrt{n})}$.

**Remark**

We don’t know whether it is possible to solve the **MAXIMUM INDEPENDENT SET problem** in time $2^{o(n)}$. In fact, we don’t expect that to happen.
Algorithm PlanarIndependentSet

Input: Plane graph $G$
Output: Maximum independent set $I$

IF $|V(G)| \leq 1$ THEN
  $I := V(G)$
ELSE
  $I := \emptyset$
  Find a $(\sqrt{8|V(G)|}, \frac{2}{3})$-separator $C$ for $G$.
  Let $A \cup B = V \setminus C$ a partition of $V$ such that
  $|A|, |B| \leq \frac{2}{3} n$, $E(A, B) = \emptyset$.
  FOR ALL independent set $S \subseteq C$ DO
    $l_A := \text{PlanarIndSet}(G[A \setminus N(S)])$
    $l_B := \text{PlanarIndSet}(G[B \setminus N(S)])$
    IF $|S| + |l_A| + |l_B| > |l|$ THEN
      $I := S \cup l_A \cup l_B$
  output $I$
Recap: Some Useful Facts About Planar Graphs

- A (simple) planar graph on \( n \geq 3 \) vertices has at most \( 3n - 6 \) edges and at most \( 2n - 4 \) faces.
- **Plane graph** = planar graph together with a specified drawing
- **Data structure**: to store a plane graph, store for each edge four pointers to its clockwise and counterclockwise neighbors at each endpoints; with each vertex, store a pointer to some incident edge.
- A **triangulation** is a simple plane graph in which every face is a triangle. Any simple plane graph \( G \) can be extended, in time \( O(n) \), to a triangulation by adding additional edges to the drawing.

- **Fact.** Planarity is preserved under *edge contractions*.
- **Corollary.** Contracting a *connected subgraph* of a plane graph to a single point preserves planarity.

**Theorem (Hopcroft–Tarjan 1974)**

Planarity of a graph can be tested, and a drawing be found, in linear time.
Spanning Trees of Small Diameter

Lemma
Let $G$ be a connected planar graph, and let $T$ be a spanning tree of $G$ with diameter $s$. Then $G$ has an $(s + 1, \frac{2}{3})$-separator. Given a plane drawing of $G$, we can find the separator in time $O(n)$.

Proof of the theorem using the lemma

- W.l.o.g., $G$ is connected.
- Fix arbitrary $v_0 \in V(G)$. Define levels

$$L_i := \{v \in V(G) : \text{dist}(v, v_0) = i\}.$$ 

Let $h := \max\{i : L_i \neq \emptyset\}$.
- If $2h + 1 \leq \sqrt{8n}$, we are done, by the lemma.
- Else, set $s := \lceil \sqrt{n/2} \rceil$ and $S_j := \bigcup_{i \equiv j \pmod{s}} L_i$.
- By averaging, there is $j_0$ with $|S_{j_0}| \leq \lfloor n/s \rfloor \leq \sqrt{2n}$ (we use $s \leq h$); remove $S_{j_0}$ from $G$. 

Proof of the theorem using the lemma, cont’d

- **Case 1**: All components of $G - S_{j_0}$ are of order $\leq \frac{2}{3} n$. We are done, $S_{j_0}$ is the desired separator.

- **Case 2**: There is one component $K$, $|K| > \frac{2}{3} n$. We have

  $$K \subseteq \bigcup_{i=j+1}^{j+s-1} L_i,$$

  for some $j \equiv j_0 \pmod{s}$

- Inside $G[K \cup L_j]$, contract $L_j$ to a single vertex. This preserves planarity (why)?

- The resulting graph $H$ has a spanning tree with diameter $2(s - 1)$, so by the lemma, we have a $(2s - 1, \frac{2}{3})$-separator $S_H$ in $H$.

- Then $S_{j_0} \cup S_H$ is the appropriate separator of $G$. 
Proof of the lemma

Lemma
Let $G$ be a plane graph and $T$ a spanning tree of $G$ with diameter $s$. Then a $(s + 1, \frac{2}{3})$-separator of $G$ can be found in time $O(n)$. 

Proof

- W.l.o.g., $G$ is a triangulation (linear time).
- For $e \in E(G) \setminus E(T)$, there is a unique cycle $C(e)$ in $T + e$, defines two regions $\text{Int}(C(e))$ and $\text{Ext}(C(e))$.
- Let $n_{\text{Int}}(e)$ be the number of vertices of $G$ in $\text{Int}(e)$, and $n_{\text{Ext}}(e)$ the number of vertices of $G$ in $\text{Ext}(e)$.
- Wanted: an edge $e$ such that both $n_{\text{Int}}(e)$ and $n_{\text{Ext}}(e)$ are $\leq \frac{2}{3} n$. Then the vertex set of $C(e)$ is the desired separator.
Proof of the lemma, cont’d

- Start with an arbitrary edge \( e = xy \in E(G) \setminus E(T) \)
- Suppose \( n_{\text{Int}}(e) > \frac{2}{3} n \) (Int and Ext interchangeable)
- Goal: Find \( e' \in E(G) \setminus E(T) \) such that
  - \( n_{\text{Ext}}(e') \leq \frac{2}{3} n \), and
  - \( \text{Int}(e') \subsetneq \text{Int}(e) \).

  If also \( n_{\text{Int}}(e') \leq \frac{2}{3} n \), we are done. Else, repeat.
- Let \( F = xyz \) be the unique (triangular) face incident to \( e = xy \) inside \( \text{Int}(C(e)) \).
  - **Case 1.** \( xz \in E(T) \). Then \( yz \notin E(T) \). Set \( e' := yz \).
  - **Case 1’** \( yz \in E(T) \) is symmetric.
  - **Case 2.** \( xz, yz \notin E(T) \). If \( n_{\text{Int}}(C(xz)) \geq n_{\text{Int}}(C(yz)) \) then choose \( e' := xz \), else \( e' := yz \). With this choice,

\[
 n_{\text{Ext}}(e') \leq n_{\text{Ext}}(e) + \frac{n_{\text{Int}}(e)}{2} + |C(e)| \leq n - \frac{n_{\text{Int}}(e)}{2} \leq \frac{2}{3} n
\]
The Algorithm

**Input:** Plane triangulation $G$, spanning tree $T \subseteq G$

**Output:** Edge $e \in E \setminus E(T)$; $C(e)$ is a separator with $n_{\text{ext}}(C(e)), n_{\text{int}}(C(e)) \leq \frac{2}{3} n$.

$e = xy \in E \setminus E(T)$ arbitrary, with direction.

Run Clockwise-DFS($y, x, x$) to determine $n_{\text{int}}(C(e))$ and $n_{\text{int}}(C(e))$.

**IF** $n_{\text{ext}}(C(e)) > \frac{2}{3} n$ **THEN**

**Update** $y := x, x := y$ \hspace{1cm} (e changes direction)

**IF** $n_{\text{int}}(C(e)) > \frac{2}{3} n$

**WHILE** $n_{\text{int}}(C(e)) > \frac{2}{3} n$ **DO**

$z \in C(e) \cup \text{Int}(C(e))$ such that $\{z, x, y\}$ is a face.

Alternately run Clockwise-DFS($z, y, x$) and Anticlockwise-DFS($z, x, y$).

**IF** Clockwise-DFS terminates first **THEN**

$n_{\text{int}}(C(zx)) \leq n_{\text{int}}(C(zy))$; **Update** $e := zy$.

ELSE

**Update** $e := zx$.

**Output** $e$.
Algorithm Clockwise-DFS: Counting $n_{\text{Int}}$ and $n_{\text{Ext}}$

Clockwise-DFS($z,y,x$)

**Input:** plane triangulation $G$, spanning tree $T \subseteq G$, cyclic lists $L_v$ of the neighbors of $v \in V$ in $G$, those which are also neighbors in $T$ are marked, root vertex $z$, reference vertex $y$, target vertex $x$

$u := y$, $v := z$.

**WHILE** $v \neq x$ **DO**

$u := v$,

$v := T$-neighbor of $v$ coming first after $u$ in $L_v$

**according to the anticlockwise direction.**

**Remark**

The tree produced by Clockwise-DFS tends to “bend” in the clockwise direction. For AntiClockwise-DFS: Replace “clockwise” with “anticlockwise”.